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Integral control of infinite-dimensional linear systems subject to input hysteresis

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Signature of Author ....

Adam D. Mawby

To Lucie, for all your love

Abstract

For both continuous-time and discrete-time, we introduce a general class of causal dynamic hysteresis nonlinearities, with certain monotonicity and Lipschitz continuity properties. It is shown that closing the loop around a stable, single-input, single-output, infinite-dimensional, linear system, subject to an input hysteresis nonlinearity from the class and compensated by an integral controller, guarantees asymptotic tracking of constant reference signals, provided that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive integrator gain is smaller than a certain constant given by a positive-real condition in terms of the linear part of the plant, and (c) the reference value is feasible in a very natural sense. The class of nonlinearities under consideration contains in particular relay hysteresis, backlash and hysteresis operators of Prandtl and Preisach type.

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Contents

Chapter 1

Introduction

Let us recall what is meant by the time-invariant, linear, finite-dimensional, single-input, single-output system (A, B, C, D) on the state-space $\mathbb{R}^n$. We have that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C^T \in \mathbb{R}^n$ and $D \in \mathbb{R}$. Moreover, the state, $x(t) \in \mathbb{R}^n$, the input, $u(t) \in \mathbb{R}$, and the output $y(t) \in \mathbb{R}$, are related by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \quad x(0) = x_0\,, \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned}$$

where $x_0 \in \mathbb{R}^n$ is an arbitrary initial condition. For a locally integrable input u, x and y are given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\,,$$

and

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)\,.$$

If we let $x_0 = 0$, then taking Laplace transforms in the above equation, we obtain the frequency-domain description:

$$\hat{y}(s) = C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s)\,,$$

which leads us to define the transfer function $\mathbf{G}(s)$ to be

$$\mathbf{G}(s) := C(sI - A)^{-1}B + D\,.$$

The closed-loop system, represented in Figure 1, can then be written

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\,, \quad x(0) = x_0 \in \mathbb{R}^n\,, && \text{(1.1a)} \\ \dot{u}(t) &= k[r - Cx(t) - Du(t)]\,, \quad u(0) = u_0 \in \mathbb{R}\,, && \text{(1.1b)} \end{aligned}$$

where $k \in \mathbb{R}$ is called the integrator gain and $r \in \mathbb{R}$ the reference value.

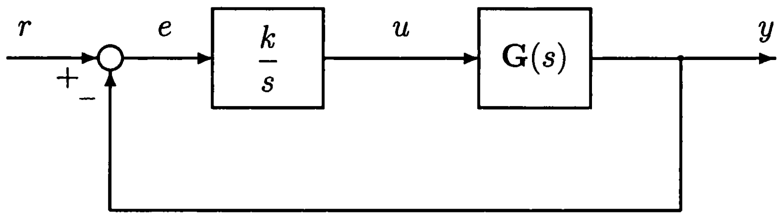

Figure 1: Low-gain integral control of finite-dimensional linear system

It has been shown by Davison [9], Lunze [29] and Morari [32] that if A is stable (that is the eigenvalues of A are in the open left-half plane), $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$, then the output $y(t)$ of the above closed-loop system, shown in Figure 1, converges to the reference value r as $t \to \infty$.[†] Additionally we will have that $x(t) \to A^{-1}Br/\mathbf{G}(0)$ and $u(t) \to r/\mathbf{G}(0)$ as $t \to \infty$.

The above result has been extended by Logemann et al. [17], Logemann and Owens [22], Logemann and Townley [28], Pohjolainen [35, 36], Pohjolainen and Lätti [37] to various classes of linear infinite-dimensional systems. A large class of linear infinite-dimensional systems can be represented in a way similar to (1.1), but the state-space will be an infinite-dimensional Hilbert space rather than the finite-dimensional vector space $\mathbb{R}^n$.

In a recent paper, Logemann, Ryan and Townley [26] have proved that the above principle remains true if the plant to be controlled is an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output system subject to a static input nonlinearity Φ (such as, for example, saturation), see Figure 2. More precisely, it is shown in [26] that if $\mathbf{G}(s)$ is the transfer function of an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output system which is such that $\mathbf{G}(0) > 0$, Φ is a static non-decreasing globally Lipschitz function, with Lipschitz constant λ, and K is the supremum of the set of all $k > 0$ such that the function

$$1 + k\,\mathrm{Re}\,\frac{\mathbf{G}(s)}{s}$$

is positive real, then for all $k \in (0, K/\lambda)$, the output $y(t)$ of the closed-loop system, shown in Figure 2, converges to r as $t \to \infty$, provided that $r/\mathbf{G}(0) \in \mathrm{clos}\,(\mathrm{im}\,\Phi)$.

There exists a substantial literature on problems related to those considered in [26], see for example Fliegner, Logemann and Ryan [10], Logemann and Curtain

[†]Therefore, under the above assumptions on the plant, the problem of tracking constant reference signals reduces to that of tuning the gain parameter k. This so-called "tuning regulator theory" [9] has been successfully applied in process control (see [6], [30]).

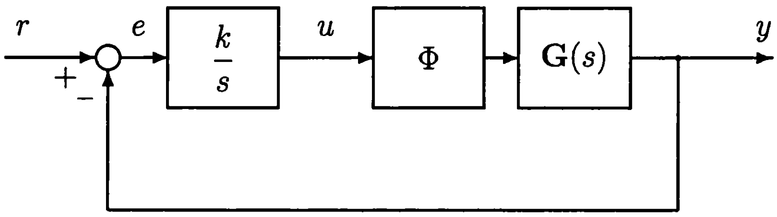

Figure 2: Low-gain control with input nonlinearity

[18], and Logemann and Ryan [23, 24]. Here we consider similar problems, but for wider classes of causal dynamic nonlinearities Φ which satisfy certain Lipschitz conditions. The classes encompass, in particular, a large number of hysteresis nonlinearities important in applications, such as relay, backlash and elastic-plastic hysteresis. Generally speaking, hysteresis is a special type of memory-based relation between a scalar input signal $u(\cdot)$ and a scalar output signal $v(\cdot)$ that cannot be expressed in terms of a single-valued function, but takes the form of "hysteresis" loops, see Figure 3. In particular, the operator $u(\cdot) \mapsto v(\cdot)$ is causal

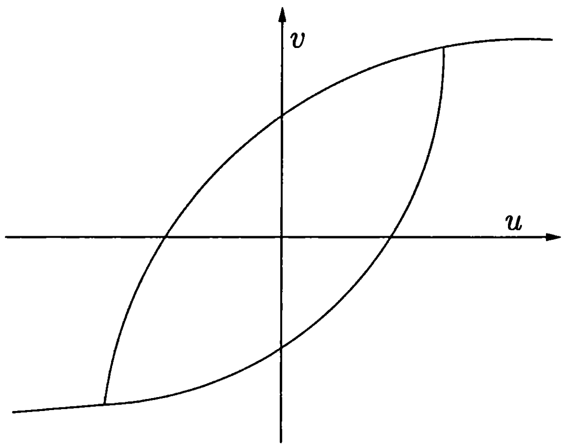

Figure 3: Hysteresis loop

and rate independent. This type of behaviour arises in mechanical plays, thermostats, elastoplasticity, ferromagnetism and in smart material structures such as piezoelectric elements and magnetostrictive transducers (see Banks et al. [1] for hysteresis phenomena in smart materials). There exists a substantial literature on mathematical modelling and mathematical theory of hysteresis phenomena, see for example Brokate [3], Brokate and Sprekels [4], Krasnosel'skiĭ and Pokrovskiĭ [16], Macki et al. [31] and Visintin [39]. Of particular importance in a systems and control context is the pioneering work [16].

We now give some additional details on the problems considered. Chapter 2 consists of some preliminaries from functional analysis. In Chapter 3 we first introduce the concept of a regular linear system. The class of regular linear systems introduced by Weiss (see [40]–[43]), is rather general and allows for highly unbounded control and observation operators. It includes most distributed parameter systems and time-delay systems of interest in control engineering. We go on to consider closed-loop systems of the form shown in Figure 2, where $\mathbf{G}(s)$ is the transfer function of a regular linear system, Φ is assumed to be a causal operator and satisfy a bounded input, bounded output condition and a weak Lipschitz condition, and the gain k can be time varying. We prove existence and uniqueness results for the closed-loop system which are very general and can be applied to the constant gain problem considered in Chapter 5 and the time-varying and adaptive gain problems considered in Chapter 9.

In Chapter 4, we introduce the concept of a hysteresis operator, based on the ideas of Brokate [3], Brokate and Sprekels [4] and Visintin [39], to be an operator which is both causal and rate indepentent. For continuous piecewise monotone input signals u this means that at time $t \in \mathbb{R}_+$, the value $v(t)$ of the output signal v is dependent only on the local extrema of u restricted to the time interval $[0, t]$. Usually, hysteresis operators are defined on spaces of continuous functions (in particular, continuous piecewise monotone functions). For certain applications such as sampled-data control of systems with hysteresis effects (see Chapter 8) it is desirable to extend hysteresis operators to spaces of piecewise continuous functions. We show that this can be done in great generality. More precisely, we show that any hysteresis operator defined on the space of continuous piecewise monotone functions, can be extended in a natural way to the space of piecewise continuous piecewise monotone functions and that this extension is a hysteresis operator. We give examples of some well known hysteresis operators, such as relay, backlash, elastic-plastic and Preisach.

In Chapter 5, we introduce three classes of hysteresis operators which we name $\mathcal{N}_c(\lambda)$, $\mathcal{N}_{sd}(\lambda)$ and $\mathcal{N}_d(\lambda)$, where λ is a weak Lipschitz constant for the operators. The three classes will be needed in later chapters. We show that the examples of hysteresis operator introduced in Chapter 4 are contained in these classes. Finally, we introduce the concept of a critical numerical value of a hysteresis operator Φ and identify the critical numerical values of some of the previously introduced hysteresis operators.

In Chapter 6, we consider the same problem as in [26] (described earlier), but for the wider class of causal dynamic nonlinearities $\mathcal{N}_c(\lambda)$. As in [26] we assume that the linear part of the system to be controlled (described in Figure 2 by the transfer function $\mathbf{G}(s)$) is an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output system. The main result in this chapter shows that for $\Phi \in \mathcal{N}_c(\lambda)$, the output $y(t)$ of the closed-loop system, shown in Figure 2, converges to r as $t \to \infty$, provided that $\mathbf{G}(0) > 0$, r is feasible in some natural sense and $k \in (0, K/\lambda)$, where K is the supremum of the set of all numbers $k > 0$ such that the function

$$1 + k \operatorname{Re} \frac{\mathbf{G}(s)}{s}$$

is positive real. We also show that so long as $r/\mathbf{G}(0)$ is not a critical numerical value of Φ then the convergence of the output y of the closed-loop system, shown in Figure 2, is of exponential order.

In Chapter 7, we provide a discrete-time analogy of the continuous-time results contained in Chapter 6. More precisely, we derive a discrete-time version of the continuous-time tuning regulator result by showing that for a power-stable, linear, infinite-dimensional, discrete-time, single-input, single-output plant with transfer function $\mathbf{G}(z)$, subject to a dynamic input nonlinearity $\Phi \in \mathcal{N}_d(\lambda)$, the output $y(n)$ of the closed-loop system, shown in Figure 4, converges to the reference value r as $n \to \infty$, provided that $\mathbf{G}(1) > 0$, r is feasible in some natural sense and $k \in (0, K/\lambda)$, where K is supremum of the set of all numbers $k > 0$ such that

$$1 + k \operatorname{Re} \frac{\mathbf{G}(z)}{z-1} \geq 0, \quad \forall |z| > 1.$$

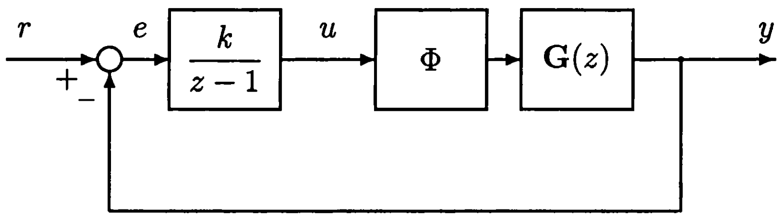

Figure 4: Low-gain control with input nonlinearity

In Chapter 8, we apply the discrete-time theory of Chapter 7 in the development of a sampled-data counterpart to the continuous-time low-gain control result of Chapter 6. Specifically, we show that for an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output plant with transfer function $\mathbf{G}(s)$, subject to a continuous-time hysteresis input nonlinearity $\tilde{\Phi}$, the output $y(t)$ of the closed-loop system, shown in Figure 5, converges to the reference value r as $t \to \infty$, provided that $\mathbf{G}(0) > 0$, r is feasible in some natural sense and $k > 0$ is sufficiently small. We remark that $\tilde{\Phi}$ is an extension of an operator $\Phi \in \mathcal{N}_{sd}(\lambda)$ and that this extension is defined in Chapter 4. In Figure 5, H denotes a standard hold operation, whilst S is a sampling operation which,

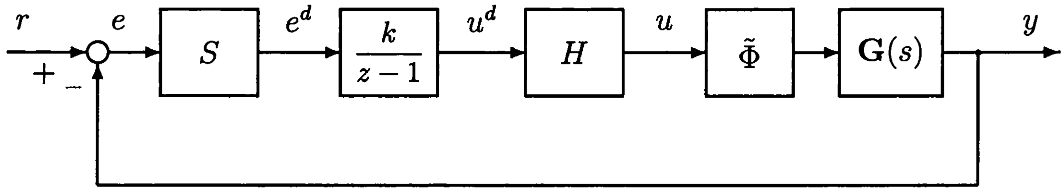

Figure 5: Sampled-data low-gain control

in the case of unbounded observation, involves an averaging operation.

In [24] it was established that an application of the simple adaptive gain strategy

$$k(t) = \frac{1}{l(t)}, \quad \text{where } \dot{l}(t) = |r - y(t)|,\ l(0) = l_0 > 0\,, \tag{1.2}$$

guarantees that the output $y(t)$ of the closed-loop system, shown in Figure 2, converges to r as $t \to \infty$ so long as Φ is a non-decreasing, globally Lipschitz function, $\mathbf{G}(0) > 0$ and $r/\mathbf{G}(0) \in \operatorname{clos}(\operatorname{im}\Phi)$. Additionally, if $r/\mathbf{G}(0)$ is not a critical value of Φ, then the gain $k(t)$ converges to a positive value as $t \to \infty$. In Chapter 9, we consider the same problem as in [24], but for the wider class of causal dynamic nonlinearities $\mathcal{N}_c(\lambda)$. That is we address aspects of adaptive tuning of the integrator gain for an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output system, subject to an input nonlinearity $\Phi \in \mathcal{N}_c(\lambda)$. In particular, we show that if the reference signal r is feasible in a natural sense and $\mathbf{G}(0) > 0$, then the adaptive gain strategy (1.2) ensures that the output $y(t)$ of the closed-loop system, shown in Figure 2, converges to r as $t \to \infty$. Additionally, we show that if $r/\mathbf{G}(0)$ is not a critical numerical value of Φ then the gain $k(t)$ converges to a positive value as $t \to \infty$. Discrete-time and sampled-data counterparts of the above continuous-time adaptive control result conclude the chapter.

Some technicalities have been relegated to the Appendices (Chapter 10).

1.1 Notation

We define

$$\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x \geq 0\}\,, \quad \mathbb{Z}_+ = \{x \in \mathbb{Z} \,|\, x \geq 0\}\,, \quad \mathbb{N} := \mathbb{Z}_+ \setminus \{0\}\,.$$

For sets M and N we denote the set of all functions $f : M \to N$ by $F(M, N)$. We define the *unit step function* $U : \mathbb{R} \to \mathbb{R}$ by $U(x) = 0$ for $x < 0$ and $U(x) = 1$ for $x \geq 0$. For $\tau \in \mathbb{R}_+$ and $n \in \mathbb{Z}_+$, the *continuous-time truncation operator* $\mathbf{P}_\tau$:

$F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is given by $(\mathbf{P}_\tau u)(t) = u(t)$ if $t \in [0, \tau]$ and $(\mathbf{P}_\tau u)(t) = 0$ otherwise, and the *discrete-time truncation operator* $\mathbf{P}_n^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is given by $(\mathbf{P}_n^d u)(k) = u(k)$ if $k \in \{0, 1, \ldots, n\}$ and $(\mathbf{P}_n^d u)(k) = 0$ otherwise. If $I \subset \mathbb{R}$ is a compact interval, then we say that a function $f \in F(I, \mathbb{R})$ is *piecewise* C^1 if there exist $\min I = x_0 < x_1 < x_2 < \ldots < x_n = \max I$ such that f is continuously differentiable on each of the intervals $[x_i, x_{i+1}]$. A function $f \in F(\mathbb{R}, \mathbb{R})$ is called piecewise C^1 if it is piecewise C^1 on any compact interval $I \subset \mathbb{R}$. As usual, for a piecewise C^1 function $f \in F(\mathbb{R}, \mathbb{R})$, we define the functions $f'_+, f'_- : \mathbb{R} \to \mathbb{R}$ by

$$f'_+ : x \mapsto \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} \qquad \text{and} \qquad f'_- : x \mapsto \lim_{h \uparrow 0} \frac{f(x+h) - f(x)}{h}.$$

If $I \subset \mathbb{R}$ is a compact interval, then $AC(I, \mathbb{R})$ denotes the space of absolutely continuous real-valued functions defined on I; $AC(\mathbb{R}_+, \mathbb{R})$ denotes the space of real-valued functions defined on $\mathbb{R}_+$ which are absolutely continuous on any compact interval $I \subset \mathbb{R}_+$, i.e. a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is in $AC(\mathbb{R}_+, \mathbb{R})$ if and only if there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(\tau)\, d\tau, \quad \forall\, t \in \mathbb{R}_+ .$$

We say that a function $f \in F(\mathbb{R}_+, \mathbb{R})$ is *piecewise monotone* if there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{i \to \infty} t_i = \infty$ and f is monotone on each of the intervals (t_i, t_{i+1}). The set of continuous piecewise monotone functions $f : \mathbb{R}_+ \to \mathbb{R}$ is denoted by $C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$. We remark that since $C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ is not closed under addition, it is not a vector space (see Appendix 1 for counterexample). A function $f \in F(\mathbb{R}_+, \mathbb{R})$ is called *piecewise continuous* if there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{i \to \infty} t_i = \infty$, f is continuous on each of the intervals (t_i, t_{i+1}) and the right and left limits of f exist and are finite at each t_i. We denote the space of all piecewise continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ by $PC(\mathbb{R}_+, \mathbb{R})$. As usual, for $f \in PC(\mathbb{R}_+, \mathbb{R})$, we define

$$f(t+) := \lim_{\tau \downarrow t} f(\tau) \ (\text{for } t \geq 0) \qquad \text{and} \qquad f(t-) := \lim_{\tau \uparrow t} f(\tau) \ (\text{for } t > 0).$$

Let $\mathbb{T} = \mathbb{R}_+, \mathbb{Z}_+$; a function $f \in F(\mathbb{T}, \mathbb{R})$ is called *ultimately constant* if there exists $T \in \mathbb{T}$ such that f is constant on $[T, \infty) \cap \mathbb{T}$.

$L(X, Y)$ denotes the space of bounded linear operators from a Banach space X to a Banach space Y and we set $L(X) := L(X, X)$. For a Banach space X, $\alpha \in \mathbb{R}$ and $\beta > 0$, we define the exponentially weighted L^p-space $L^p_\alpha(\mathbb{R}_+, X) := \{f \in L^p_{\text{loc}}(\mathbb{R}_+, X) \mid f(\cdot) \exp(-\alpha \cdot) \in L^p(\mathbb{R}_+, X)\}$ and the weighted l^p-space $l^p_\beta(\mathbb{Z}_+, X) :=$

$\{f \in l^p_{\text{loc}}(\mathbb{Z}_+, X) \mid f(\cdot)\beta^{-\cdot} \in l^p(\mathbb{Z}_+, X)\}$. For $\alpha \in \mathbb{R}$ and $\beta > 0$ define, $\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$ and $\mathbb{E}_\beta := \{z \in \mathbb{C} \mid |z| > \beta\}$. Moreover, set

$$\begin{aligned} H^\infty(\mathbb{C}_\alpha) &:= \{f : \mathbb{C}_\alpha \to \mathbb{C} \mid f \text{ is holomorphic and bounded}\}, \\ H^\infty(\mathbb{E}_\beta) &:= \{f : \mathbb{E}_\beta \to \mathbb{C} \mid f \text{ is holomorphic and bounded}\}. \end{aligned}$$

The Laplace transform is denoted by $\mathfrak{L}$ and the z-transform by $\mathfrak{Z}$. The set of all signed Borel measures on $\mathbb{R}_+$ is denoted by $\mathcal{M}(\mathbb{R}_+)$. For $\mu \in \mathcal{M}(\mathbb{R}_+)$, $|\mu|$ denotes the total variation of μ. We denote the Lebesgue measure on $\mathbb{R}_+$ by μ_L. We denote the indicator function of the set S by χ_S.

Chapter 2

Preliminaries from functional analysis

Let $C(\mathbb{R}_+, \mathbb{R})$ denote the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$. We want to define a concept of convergence in the space $C(\mathbb{R}_+, \mathbb{R})$ and therefore we introduce a topology on $C(\mathbb{R}_+, \mathbb{R})$. To this end we define a family $\mathcal{U}$ of subsets of $C(\mathbb{R}_+, \mathbb{R})$ by

$$\mathcal{U} := \{B(f, \varepsilon) \,|\, f \in C(\mathbb{R}_+, \mathbb{R}),\ \varepsilon > 0\},$$

where

$$B(f, \varepsilon) := \{g \in C(\mathbb{R}_+, \mathbb{R}) \,|\, \sup_{t \in \mathbb{R}_+} |f(t) - g(t)| < \varepsilon\}.$$

We endow $C(\mathbb{R}_+, \mathbb{R})$ with the topology generated by $\mathcal{U}$ (cf. [11], pp. 114–115, p. 133). This topology is called the *topology of uniform convergence* . Clearly, a sequence $(f_n) \subset C(\mathbb{R}_+, \mathbb{R})$ converges to $f \in C(\mathbb{R}_+, \mathbb{R})$ in this topology if and only if for all $\varepsilon > 0$ there exists $N > 0$ such that

$$\sup_{t \in \mathbb{R}_+} |f(t) - f_n(t)| < \varepsilon, \quad \forall n \geq N,$$

and we say that f_n *converges uniformly* to f as $n \to \infty$. We write $f_n \overset{\text{uc}}{\to} f$ as $n \to \infty$. We call a sequence $(f_n) \subset C(\mathbb{R}_+, \mathbb{R})$ a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N > 0$ such that

$$\sup_{t \in \mathbb{R}_+} |f_m(t) - f_n(t)| < \varepsilon, \quad \forall m, n \geq N.$$

Lemma 2.1.1 *The space $C(\mathbb{R}_+, \mathbb{R})$ is complete in the sense that for each Cauchy sequence $(f_n) \subset C(\mathbb{R}_+, \mathbb{R})$, there exists $f \in C(\mathbb{R}_+, \mathbb{R})$ such that $f_n \overset{\text{uc}}{\to} f$ as $n \to \infty$.*

Proof: Let $(f_n) \subset C(\mathbb{R}_+, \mathbb{R})$ be a Cauchy sequence. For each $t \in \mathbb{R}_+$, $(f_n(t)) \subset \mathbb{R}$ is a Cauchy sequence and since $\mathbb{R}$ is complete, there exists $f_t \in \mathbb{R}$ such that $\lim_{n\to\infty} f_n(t) = f_t$. Define $f : \mathbb{R}_+ \to \mathbb{R}$ by $f(t) = f_t$ for all $t \in \mathbb{R}_+$. Let $\varepsilon > 0$ and choose $n_0 > 0$ such that

$$\sup_{t\in\mathbb{R}_+} |f_m(t) - f_n(t)| \leq \varepsilon/2\,, \quad \forall\, m, n \geq n_0\,.$$

For each $t \in \mathbb{R}_+$, there exists $n_t \geq n_0$ such that

$$|f_{n_t}(t) - f(t)| \leq \varepsilon/2\,.$$

Then for all $t \in \mathbb{R}_+$ and all $n \geq n_0$

$$|f_n(t) - f(t)| \leq |f_{n_t}(t) - f(t)| + |f_{n_t}(t) - f_n(t)| \leq \varepsilon/2 + \varepsilon/2\,.$$

Thus

$$\sup_{t\in\mathbb{R}_+} |f_n(t) - f(t)| \leq \varepsilon\,, \quad \forall\, n \geq n_0\,. \tag{2.1}$$

It remains only to show that $f \in C(\mathbb{R}_+, \mathbb{R})$. Let $t \in \mathbb{R}_+$ and $\varepsilon > 0$. By (2.1) we know that there exists $n \in \mathbb{Z}_+$ such that

$$\sup_{t\in\mathbb{R}_+} |f(t) - f_n(t)| \leq \varepsilon/3\,.$$

By continuity of f_n there exists $\delta > 0$ such that for $\tau \in \mathbb{R}_+$ with $|t - \tau| \leq \delta$

$$|f_n(t) - f_n(\tau)| \leq \varepsilon/3\,.$$

Hence, for all $\tau \in \mathbb{R}_+$ with $|t - \tau| \leq \delta$

$$|f(t)-f(\tau)| \leq |f(t)-f_n(t)|+|f_n(t)-f_n(\tau)|+|f_n(\tau)-f(\tau)| \leq \varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon\,.$$

□

Lemma 2.1.2 $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ *is dense in* $C(\mathbb{R}_+, \mathbb{R})$ *in the sense that for all* $f \in C(\mathbb{R}_+, \mathbb{R})$*, there exists a sequence* $(f_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ *such that* $f_n \overset{\mathrm{uc}}{\to} f$ *as* $n \to \infty$.

Proof: Let $f \in C(\mathbb{R}_+, \mathbb{R})$ and $\varepsilon > 0$. We proceed by defining a function $g \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that

$$|f(t) - g(t)| \leq \varepsilon\,, \quad \forall\, t \in \mathbb{R}_+\,.$$

On each $I_n := [n, n+1]$ $(n \in \mathbb{Z}_+)$, f is uniformly continuous and therefore there exists $\delta_n > 0$ such that for all $t_1, t_2 \in I_n$

$$|t_1 - t_2| < \delta_n \quad \Longrightarrow \quad |f(t_1) - f(t_2)| < \varepsilon/2\,. \tag{2.2}$$

Choose $K_n \in \mathbb{Z}_+$ such that $K_n > 1/\delta_n$. Define g on each I_n as follows: let $g(n + k/K_n) = f(n + k/K_n)$ for all $k = 0, 1, \ldots, K_n$ and let g be affine linear on each $[n + k/K_n, n + (k+1)/K_n]$ $(k = 0, 1, \ldots, K_n - 1)$. It is clear that g is continuous and piecewise affine linear and hence $g \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$.

Let $t \in \mathbb{R}_+$, then there exists $n \in \mathbb{Z}_+$ and $k \in \{0, 1, \ldots, K_n - 1\}$ such that $t \in [n + k/K_n, n + (k+1)/K_n)$. Thus by (2.2)

$$\begin{aligned}
|f(t) - g(t)| &\leq |f(t) - f(n + k/K_n)| + |g(t) - g(n + k/K_n)| \\
&\leq |f(t) - f(n + k/K_n)| + |g(n + (k+1)/K_n) - g(n + k/K_n)| \\
&= |f(t) - f(n + k/K_n)| + |f(n + (k+1)/K_n) - f(n + k/K_n)| \\
&< \varepsilon/2 + \varepsilon/2\,.
\end{aligned}$$

□

We introduce a concept of Lipschitz continuity for operators on $C(\mathbb{R}_+, \mathbb{R})$ or $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and then show that any Lipschitz continuous operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ can be extended to a unique Lipschitz continuous operator on $C(\mathbb{R}_+, \mathbb{R})$.

Definition 2.1.3 Let $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. An operator $\Phi : \mathcal{C} \to F(\mathbb{R}_+, \mathbb{R})$ is called *Lipschitz continuous* with *Lipschitz constant* $l > 0$ if

$$\sup_{t \in \mathbb{R}_+} |(\Phi(f))(t) - (\Phi(g))(t)| \leq l \sup_{t \in \mathbb{R}_+} |f(t) - g(t)|\,, \quad \forall\, f, g \in \mathcal{C}\,.$$

◇

Lemma 2.1.4 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be a Lipschitz continuous operator with Lipschitz constant $\lambda > 0$. Then there exists a unique Lipschitz continuous extension $\Phi_e : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ of Φ with Lipschitz constant λ.*

Proof: Let $f \in C(\mathbb{R}_+, \mathbb{R})$. By Lemma 2.1.2 there exists $(f_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $f_n \overset{\mathrm{uc}}{\to} f$ as $n \to \infty$. We observe that $(\Phi(f_n))$ is a Cauchy sequence, since for $m, n \in \mathbb{Z}_+$, $\sup_{t \in \mathbb{R}_+} |(\Phi(f_m))(t) - (\Phi(f_n))(t)| \leq \lambda \sup_{t \in \mathbb{R}_+} |f_m(t) - f_n(t)|$ and (f_n) is a Cauchy sequence. Since, by Lemma 2.1.1, $C(\mathbb{R}_+, \mathbb{R})$ is complete, there exists $\tilde{f} \in C(\mathbb{R}_+, \mathbb{R})$ such that $\Phi(f_n) \overset{\mathrm{uc}}{\to} \tilde{f}$ as $n \to \infty$. To see that $\tilde{f}$ does not depend upon the choice of sequence (f_n), let $(g_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ be another sequence such that $g_n \overset{\mathrm{uc}}{\to} f$ as $n \to \infty$. Then for all $n \in \mathbb{Z}_+$

$$\begin{aligned}\sup_{t\in\mathbb{R}_+} |(\Phi(f_n))(t) - (\Phi(g_n))(t)| &\leq \lambda \sup_{t\in\mathbb{R}_+} |f_n(t) - g_n(t)| \\ &\leq \lambda \left[\sup_{t\in\mathbb{R}_+} |f(t) - f_n(t)| + \sup_{t\in\mathbb{R}_+} |f(t) - g_n(t)|\right].\end{aligned}$$

Consequently the sequences $(\Phi(f_n))$ and $(\Phi(g_n))$ have the same limit. Thus, setting $\Phi_e(f) := \tilde{f}$ for all $f \in C(\mathbb{R}_+, \mathbb{R})$, makes Φ_e a well-defined extension of Φ to $C(\mathbb{R}_+, \mathbb{R})$.

To show that $\Phi_e : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is a Lipschitz continuous operator with Lipschitz constant λ, observe that for $f, g \in C(\mathbb{R}_+, \mathbb{R})$ and $(f_n), (g_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $f_n \overset{\mathrm{uc}}{\to} f$ and $g_n \overset{\mathrm{uc}}{\to} g$ as $n \to \infty$, it follows that for $n \in \mathbb{Z}_+$

$$\begin{aligned}&\sup_{t\in\mathbb{R}_+} |(\Phi_e(f))(t) - (\Phi_e(g))(t)| \\ &\quad\leq \sup_{t\in\mathbb{R}_+} [|(\Phi_e(f))(t) - (\Phi(f_n))(t)| + |(\Phi_e(g))(t) - (\Phi(g_n))(t)| \\ &\qquad\quad + |(\Phi(f_n))(t) - (\Phi(g_n))(t)|] \\ &\quad\leq \sup_{t\in\mathbb{R}_+} |(\Phi_e(f))(t) - (\Phi(f_n))(t)| + \sup_{t\in\mathbb{R}_+} |(\Phi_e(g))(t) - (\Phi(g_n))(t)| \\ &\qquad\quad + \lambda \sup_{t\in\mathbb{R}_+} |f_n(t) - g_n(t)|.\end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$\sup_{t\in\mathbb{R}_+} |(\Phi_e(f))(t) - (\Phi_e(g))(t)| \leq \lambda \sup_{t\in\mathbb{R}_+} |f(t) - g(t)|.$$

□

Chapter 3

Regular infinite-dimensional linear systems with nonlinear feedback

3.1 Preliminaries

We assemble some fundamental facts pertaining to regular linear systems and tailored to later requirements: the reader is referred to [40]–[43] for full details. This section is prefaced with the remark that the class of regular linear infinite-dimensional systems is rather general: it includes most distributed parameter systems and all time-delay systems (retarded and neutral) which are of interest in applications. Although there exist abstract examples of well-posed, infinite-dimensional systems that fail to be regular, the author is of the opinion that any physically-motivated, well-posed, linear, continuous-time, autonomous control system is regular.

First, some notation: for any Hilbert space H and any $\tau \geq 0$, we define the *right-shift* operator $\mathbf{R}_\tau : L^2_{\mathrm{loc}}(\mathbb{R}_+, H) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, H)$, by

$$(\mathbf{R}_\tau(u))(t) = \begin{cases} 0 & \text{if } t \in [0, \tau)\,, \\ u(t-\tau) & \text{if } t \geq \tau\,. \end{cases}$$

Well-posed systems

For $u, v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H)$ and $\tau \in \mathbb{R}_+$, the *τ-concatenation* $u \overset{\tau}{\diamond} v$ is defined by

$$u \overset{\tau}{\diamond} v = \mathbf{P}_\tau u + \mathbf{R}_\tau v\,.$$

The following concept was introduced by Weiss [43]. An equivalent definition can be found in Salamon [38].

Definition 3.1.1 Let X be a real Hilbert space. A *well-posed linear system* with *state-space* X, *input-space* $\mathbb{R}$ and *output-space* $\mathbb{R}$ is the quadruple $\mathbf{\Sigma} = (\mathbf{T}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$, where

(1) $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X,

(2) $\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, \mathbb{R})$ to X such that

$$\mathbf{\Phi}_{\tau+t}(u \overset{\tau}{\diamond} v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v\,,$$

for all $u, v \in L^2(\mathbb{R}_+, \mathbb{R})$ and all $\tau, t \in \mathbb{R}_+$,

(3) $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^2(\mathbb{R}_+, \mathbb{R})$ such that

$$\mathbf{\Psi}_{\tau+t} x_0 = \mathbf{\Psi}_\tau x_0 \overset{\tau}{\diamond} \mathbf{\Psi}_t \mathbf{T}_\tau x_0\,,$$

for all $x_0 \in X$ and all $\tau, t \in \mathbb{R}_+$, and $\mathbf{\Psi}_0 = 0$,

(4) $\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^2(\mathbb{R}_+, \mathbb{R})$ to $L^2(\mathbb{R}_+, \mathbb{R})$ such that

$$\mathbf{F}_{\tau+t}(u \overset{\tau}{\diamond} v) = \mathbf{F}_\tau u \overset{\tau}{\diamond} (\mathbf{\Psi}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v)\,,$$

$u, v \in L^2(\mathbb{R}_+, \mathbb{R})$ and all $\tau, t \in \mathbb{R}_+$, and $\mathbf{F}_0 = 0$.

$\diamond$

For an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and an initial state $x_0 \in X$, the associated state function $x \in C(\mathbb{R}_+, X)$ and output function $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ of $\mathbf{\Sigma}$ are given by

$$x(t) = \mathbf{T}_t x_0 + \mathbf{\Phi}_t \mathbf{P}_t u\,, \tag{3.1a}$$

$$\mathbf{P}_t y = \mathbf{\Psi}_t x_0 + \mathbf{F}_t \mathbf{P}_t u\,. \tag{3.1b}$$

We say that $\mathbf{\Sigma}$ is *exponentially stable* if the semigroup $\mathbf{T}$ is exponentially stable, i.e.

$$\omega(\mathbf{T}) := \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{T}_t\| < 0\,,$$

where $\omega(\mathbf{T})$ is called the *exponential growth constant* of $\mathbf{T}$. It is clear that there exist unique operators $\mathbf{\Psi}_\infty : X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and $\mathbf{F}_\infty : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ such that for all $\tau \geq 0$

$$\mathbf{\Psi}_\tau = \mathbf{P}_\tau \mathbf{\Psi}_\infty\,, \quad \mathbf{F}_\tau = \mathbf{P}_\tau \mathbf{F}_\infty\,.$$

We call $\mathbf{\Psi}_\infty$ the *state-to-output map* and $\mathbf{F}_\infty$ the *input-output operator.* If $\mathbf{\Sigma}$ is exponentially stable, then the operators $\mathbf{\Phi}_t$ and $\mathbf{\Psi}_t$ are uniformly bounded, $\mathbf{\Phi}$ is a bounded operator from X into $L^2(\mathbb{R}_+, \mathbb{R})$ and $\mathbf{F}_\infty$ maps $L^2(\mathbb{R}_+, \mathbb{R})$ boundedly into $L^2(\mathbb{R}_+, \mathbb{R})$. Since $\mathbf{P}_\tau \mathbf{F}_\infty = \mathbf{P}_\tau \mathbf{F}_\infty \mathbf{P}_\tau$ for all $\tau \in \mathbb{R}_+$, $\mathbf{F}_\infty$ is a causal operator.

It can be shown (see Weiss [40]) that if $\alpha > \omega(\mathbf{T})$ and if $u \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$, then $\mathbf{F}_\infty u \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ and there exists a unique holomorphic function $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathbb{C}$ such that

$$\mathbf{G}(s)(\mathfrak{L}u)(s) = [\mathfrak{L}(\mathbf{F}_\infty u)](s) \,, \quad \forall\, s \in \mathbb{C}_\alpha \,.$$

In particular, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for all $\alpha > \omega(\mathbf{T})$. The function $\mathbf{G}$ is called the *transfer function* of $\mathbf{\Sigma}$.

Regularity

$\mathbf{\Sigma}$ and its transfer function $\mathbf{G}(s)$ are called *regular* if the limit

$$\lim_{s \to \infty,\, s \in \mathbb{R}} \mathbf{G}(s) =: D$$

exists. The operator D is called the *feedthrough operator* of $\mathbf{\Sigma}$. The regular system is said to be *exponentially stable* if $\mathbf{\Sigma}$ is exponentially stable.

Generating operators

The generator of $\mathbf{T}$ is denoted by A. Let X_1 be the space $\operatorname{dom}(A)$ endowed with the graph norm, and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(s_0 I - A)^{-1} x\|$, where s_0 is any fixed element in $\varrho(A)$, the resolvent set of A. We have $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ can be restricted to a C_0-semigroup on X_1 and extended to a C_0-semigroup on X_{-1}. The exponential growth constant is the same on all three spaces. The generator on X_1 is the restriction of A to $\operatorname{dom}(A^2)$ and the generator on X_{-1} is an extension of A to X (which is bounded as an operator from X to X_{-1}). We shall use the same symbols for the original semigroup and its generator and the corresponding restrictions and extensions.

By a representation theorem due to Salamon [38] (see also Weiss [41, 42]) there exist unique operators $B \in L(\mathbb{R}, X_{-1})$ and $C \in L(X_1, \mathbb{R})$ (the *control operator* and the *observation operator* of $\mathbf{\Sigma}$, respectively) such that for all $t \in \mathbb{R}_+$, all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and all $x_0 \in X_1$

$$\mathbf{\Phi}_t \mathbf{P}_t u = \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \quad \text{and} \quad (\mathbf{\Psi}_\infty x_0)(t) = C \mathbf{T}_t x_0 \,.$$

B is called *bounded* if $B \in L(\mathbb{R}, X)$ (and *unbounded* otherwise), whereas C is called *bounded* if it can be extended continuously to X (and *unbounded* otherwise).

It is clear that $B \in L(\mathbb{R}, X_{-1})$ is an *admissible control operator* for $\mathbf{T}$, in the sense that for any $\tau \in \mathbb{R}_+$, the operator

$$L^2(\mathbb{R}_+, \mathbb{R}) \to X_{-1}, \qquad u \mapsto \int_0^\tau \mathbf{T}_{\tau-t} B u(t)\, dt, \tag{3.2}$$

has its range in X and $C \in L(X_1, \mathbb{R})$ is an *admissible observation operator* for $\mathbf{T}$, in the sense that for any $\tau \in \mathbb{R}_+$, the operator

$$X_1 \to L^2(\mathbb{R}_+, \mathbb{R}), \qquad x_0 \mapsto \mathbf{P}_\tau C \mathbf{T}_t x_0, \tag{3.3}$$

has a continuous extension to X.

The *Lebesgue extension* of C was introduced in [42] and is defined by

$$C_L x_0 = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbf{T}_\tau x_0\, d\tau, \tag{3.4}$$

where $\operatorname{dom}(C_L)$ is equal to the set of all those $x_0 \in X$ for which the above limit exists. Clearly $X_1 \subset \operatorname{dom}(C_L) \subset X$, and for any $x_0 \in X$ we have that $\mathbf{T}_t x_0 \in \operatorname{dom}(C_L)$ for almost every (a.e.) $t \in \mathbb{R}_+$. Furthermore,

$$(\mathbf{\Psi}_\infty x_0)(t) = C_L \mathbf{T}_t x_0 \quad \text{a.e. } t \in \mathbb{R}_+ . \tag{3.5}$$

If $\mathbf{T}$ is exponentially stable, then there exist constants $\gamma_1, \gamma_2 > 0$ such that, for all $t \in \mathbb{R}_+$, $u \in L^2(\mathbb{R}_+, \mathbb{R})$ and $x_0 \in X$,

$$\|\mathbf{\Phi}_t \mathbf{P}_t u\| = \left\| \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau \right\| \le \gamma_1 \|u\|_{L^2(\mathbb{R}_+,\mathbb{R})},$$

$$\|\mathbf{P}_t \mathbf{\Psi}_\infty x_0\|_{L^2(\mathbb{R}_+,\mathbb{R})} = \left(\int_0^t \|C_L \mathbf{T}_\tau x_0\|^2 d\tau \right)^{1/2} \le \gamma_2 \|x_0\| . \tag{3.6}$$

If $\mathbf{\Sigma}$ is regular, then for any $x_0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$, the functions $x(\cdot)$ and $y(\cdot)$, defined by (3.1), satisfy the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{3.7a}$$

$$y(t) = C_L x(t) + Du(t), \tag{3.7b}$$

for almost all $t \in \mathbb{R}_+$ (in particular, $x(t) \in \operatorname{dom}(C_L)$ for almost all $t \in \mathbb{R}_+$). The derivative on the left-hand side of (3.7a) has, of course, to be understood in X_{-1}. In other words, if we consider the initial-value problem (3.7a) in the space X_{-1}, then for any $x_0 \in X$ and $u \in L^2_{\rm loc}(\mathbb{R}_+, \mathbb{R})$, (3.7a) has unique strong solution (in the sense of Pazy [34], p. 109) given by the variation of parameters formula

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B u(\tau)\, d\tau\,. \tag{3.8}$$

It has been demonstrated in [40] that, if $\mathbf{\Sigma}$ is regular, then $(sI - A)^{-1} B\mathbb{R} \subset \mathrm{dom}(C_L)$ for all $s \in \varrho(A)$ and the transfer function $\mathbf{G}(s)$ can be expressed as

$$\mathbf{G}(s) = C_L(sI - A)^{-1}B + D\,, \quad \forall\, s \in \mathbb{C}_{\omega(\mathbf{T})}\,,$$

which is familiar from finite-dimensional system theory. The operators A, B, C and D are called the *generating operators* of $\mathbf{\Sigma}$.

Definition 3.1.2 Let $\mathfrak{L}$ denote the class of quadruples (A, B, C, D) which are the generating operators of a regular linear system $\mathbf{\Sigma}$, with state space X, input space $\mathbb{R}$, output space $\mathbb{R}$ and transfer function $\mathbf{G}(s)$, satisfying:

(a) $\mathbf{\Sigma}$ is exponentially stable; (b) $\mathbf{G}(0) > 0$.

$\diamond$

Proposition 3.1.3 *If $(A, B, C, D) \in \mathfrak{L}$, then $(A+\varepsilon I, B, C, D) \in \mathfrak{L}$ for all $\varepsilon > 0$ sufficiently small.*

Proof: Clearly, for sufficiently small $\varepsilon > 0$, $e^{\varepsilon t}\mathbf{T}_t$ is an exponentially stable semigroup. Since $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha)$ for some $\alpha < 0$, $\mathbf{G}$ is continuous at 0 and therefore for sufficiently small $\varepsilon > 0$, $C_L(-\varepsilon I - A)^{-1}B + D = \mathbf{G}(-\varepsilon) > 0$. It is clear that for all $\varepsilon > 0$, $(A+\varepsilon I, B, C, D)$ are the generating operators of a regular system with transfer function $s \mapsto \mathbf{G}(s - \varepsilon)$ and therefore for sufficiently small $\varepsilon > 0$, $(A + \varepsilon I, B, C, D) \in \mathfrak{L}$. $\square$

For future reference we state the following two lemmas. The proof of the following lemma can be found in [26] (see Lemma 2.2 in [26]) and [18] (see Lemma 2.2 in [18]).

Lemma 3.1.4 *Assume that A generates an exponentially stable semigroup $\mathbf{T}$ on a real Hilbert space X and that B is an admissible control operator for $\mathbf{T}$. Then the following statements hold.*

(1) If $u \in L^\infty(\mathbb{R}_+, \mathbb{R})$ is such that $\lim_{t\to\infty} u(t) = u_\infty$ exists, then, for all $x_0 \in X$, the state $x(\cdot)$ given by (3.8) satisfies

$$\lim_{t\to\infty} \|x(t) + A^{-1}Bu_\infty\| = 0\,.$$

(2) *There exist constants* $\alpha_0, \alpha_1 > 0$ *such that, for all* $(x_0, u) \in X \times L^2(\mathbb{R}_+, \mathbb{R})$, *the state* $x(\cdot)$ *given by (3.8) satisfies*

$$\begin{aligned} \lim_{t\to\infty} \|x(t)\| &= 0, \quad x \in L^2(\mathbb{R}_+, X), \\ \|x\|_{L^2(\mathbb{R}_+,X)} &\leq \alpha_0\|x_0\| + \alpha_1\|u\|_{L^2(\mathbb{R}_+,\mathbb{R})}. \end{aligned}$$

(3) *For all* $(x_0, u) \in X \times L^\infty(\mathbb{R}_+, \mathbb{R})$, *the state* $x(\cdot)$ *given by (3.8) satisfies*

$$x \in L^\infty(\mathbb{R}_+, X).$$

The proof of the following lemma can be found in [18] (see Lemma 2.1 in [18]).

Lemma 3.1.5 *Let* $(A, B, C, D) \in \mathfrak{L}$. *If* $u \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ *and* $u_\infty \in \mathbb{R}$ *are such that* $u - u_\infty \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ *for some* $\alpha > \omega(\mathbf{T})$, *then, for all* $x_0 \in X$, *the output* $y(\cdot)$ *given by (3.7) satisfies*

$$y - \mathbf{G}(0)u_\infty \in L^2_\alpha(\mathbb{R}_+, \mathbb{R}).$$

3.2 Existence and uniqueness of solutions for regular systems with nonlinear feedback

Let $n \in \mathbb{N}$ and $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{F} \neq \emptyset$. We call an operator $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R}^n)$ *causal* if for all $u, v \in \mathcal{F}$ and all $\tau \in \mathbb{R}_+$ with $u(t) = v(t)$ for all $t \in [0, \tau]$ it follows that $(\Phi(u))(t) = (\Phi(v))(t)$ for all $t \in [0, \tau]$.

For $\alpha \geq 0$, $w \in C([0, \alpha], \mathbb{R}^n)$ and $\delta_1, \delta_2 > 0$, we define $C(w; \delta_1, \delta_2)$ to be the set of all $u \in C(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$u(t) = w(t), \ \forall\, t \in [0, \alpha] \qquad \text{and} \qquad \|u(t) - w(\alpha)\| \leq \delta_1, \ \forall\, t \in [\alpha, \alpha + \delta_2].$$

We study an abstract Volterra integro-differential equation. Let $\alpha \geq 0$ and let $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$. Consider the initial-value problem

$$\dot{w}(t) = (Vw)(t), \quad t \geq \alpha, \tag{3.10a}$$

$$w(t) = w_\alpha(t), \quad t \in [0, \alpha], \tag{3.10b}$$

where the operator $V : C(\mathbb{R}_+, \mathbb{R}^n) \to L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is causal and *weakly locally Lipschitz* in the following sense: for all $\alpha \geq 0$ and $w \in C([0, \alpha], \mathbb{R}^n)$, there exist

$\delta > 0$, $\rho > 0$ and a continuous function $f : [0, \delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Vu)(t) - (Vv)(t)\| \, dt \le f(\varepsilon) \sup_{\alpha \le t \le \alpha+\varepsilon} \|u(t) - v(t)\| ,$$

for all $\varepsilon \in [0, \delta]$ and all $u, v \in C(w; \rho, \delta)$.

A *solution* of the initial-value problem (3.10) on an interval $[0, \beta)$, where $\beta > \alpha$, is a function $w \in C([0, \beta), \mathbb{R}^n)$, with $w(t) = w_\alpha(t)$ for all $t \in [0, \alpha]$, such that w is absolutely continuous on $[\alpha, \beta)$ and (3.10a) is satisfied for a.e. $t \in [\alpha, \beta)$.

Strictly speaking, to make sense of (3.10), we have to give a meaning to $(Vw)(t)$, $t \in [0, \beta)$, when w is a continuous function defined on a *finite* interval $[0, \beta)$ (recall that V operates on the space of continuous functions defined on the *infinite* interval $\mathbb{R}_+$). This can be easily done using causality of V: for all $t \in [0, \beta)$, $(Vw)(t) := (Vw^*)(t)$, where $w^* : \mathbb{R}_+ \to \mathbb{R}^n$ is any continuous function with $w^*(s) = w(s)$ for all $s \in [0, t]$.

Proposition 3.2.1 *For every $\alpha \ge 0$ and every $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$, there exists $\varepsilon > 0$ and a unique solution $w(\cdot)$ of (3.10) defined on $[0, \alpha + \varepsilon)$.*

Proof: Fix $\alpha \ge 0$ and $w_\alpha \in C([0, \alpha], \mathbb{R}^n)$ arbitrarily. Define a continuous extension $w_\alpha^* : \mathbb{R}_+ \to \mathbb{R}^n$ of w_α by setting $w_\alpha^*(t) = w_\alpha(\alpha)$ for all $t \in [\alpha, \infty)$. For later convenience, we introduce the continuous function

$$\varepsilon \mapsto g(\varepsilon) := \int_\alpha^{\alpha+\varepsilon} \|(Vw_\alpha^*)(t)\| \, dt .$$

Since V is weakly Lipschitz there exist $\delta > 0$, $\rho > 0$ and a continuous function $f : [0, \delta] \to \mathbb{R}_+$, with $f(0) = 0$, such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Vv)(t) - (Vw)(t)\| \, dt \le f(\varepsilon) \sup_{\alpha \le t \le \alpha+\varepsilon} \|v(t) - w(t)\| ,$$

for all $\varepsilon \in [0, \delta]$ and all $v, w \in C(w_\alpha; \rho, \varepsilon)$.

For $\varepsilon > 0$ set

$$C_\varepsilon := \{ w \in C([0, \alpha + \varepsilon], \mathbb{R}^n) \mid w(t) = w_\alpha(t) \text{ if } t \in [0, \alpha] ; \\ \|w(t) - w_\alpha(\alpha)\| \le \rho \text{ if } t \in [\alpha, \alpha + \varepsilon] \} , \quad (3.11)$$

which, endowed with the metric

$$(v, w) \mapsto \sup_{\alpha \le t \le \alpha+\varepsilon} \|v(t) - w(t)\| ,$$

is a complete metric space.

Existence and uniqueness of solutions on a small interval is proved by showing that

$$(\Gamma w)(t) := \begin{cases} w_\alpha(t)\,, & 0 \le t \le \alpha\,, \\ w_\alpha(\alpha) + \displaystyle\int_\alpha^t (Vw)(\tau)\,d\tau\,, & \alpha \le t \le \alpha + \varepsilon\,, \end{cases}$$

defines a contraction on C_ε for sufficiently small $\varepsilon > 0$.

By the weak Lipschitz property and causality of V, for all $\varepsilon \in (0, \delta)$, all $v, w \in C_\varepsilon$ and all $t \in [\alpha, \alpha + \varepsilon]$

$$\begin{aligned} \|(\Gamma w)(t) - w_\alpha(\alpha)\| &\le \int_\alpha^{\alpha+\varepsilon} \|(Vw)(\tau)\|\,d\tau \\ &\le g(\varepsilon) + \int_\alpha^{\alpha+\varepsilon} \|(Vw)(\tau) - (Vw_\alpha^*)(\tau)\|\,d\tau \\ &\le g(\varepsilon) + \rho f(\varepsilon) \\ &\le \rho \qquad \text{for sufficiently small } \varepsilon > 0 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \|(\Gamma v)(t) - (\Gamma w)(t)\| &\le \int_\alpha^{\alpha+\varepsilon} \|(Vv)(\tau) - (Vw)(\tau)\|\,d\tau \\ &\le f(\varepsilon) \sup_{\alpha \le \tau \le \alpha+\varepsilon} \|v(\tau) - w(\tau)\| \\ &\le \frac{1}{2} \sup_{\alpha \le \tau \le \alpha+\varepsilon} \|v(\tau) - w(\tau)\| \quad \text{for suff. small } \varepsilon > 0. \end{aligned} \tag{3.13}$$

By (3.12), $\Gamma(C_\varepsilon) \subset C_\varepsilon$ for all sufficiently small $\varepsilon > 0$. Consequently, we obtain from (3.13) that Γ is a contraction on C_ε for all sufficiently small $\varepsilon > 0$. □

Definition 3.2.2 Let $a \in (0, \infty]$ and let $J \subset \mathbb{R}_+$ be an interval of the form $[0, a)$ or $[0, a]$. For $\tau \in J$, we define the operator $\mathbf{Q}_\tau : F(J, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(\mathbf{Q}_\tau u)(t) = \begin{cases} u(t) & \text{for } 0 \le t \le \tau\,, \\ u(\tau) & \text{for } t > \tau\,. \end{cases}$$

◇

If the domain space of $\mathbf{Q}_\tau$ is $F(\mathbb{R}_+, \mathbb{R})$ (i.e. $J = [0, \infty)$), then $\mathbf{Q}_\tau$ is a projection operator. Given an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ and a number $a > 0$ we define an operator $\tilde{\Phi} : C([0, a), \mathbb{R}) \to F([0, a), \mathbb{R})$ by setting

$$(\tilde{\Phi}(u))(t) = (\Phi(\mathbf{Q}_t u))(t)\,, \quad \forall\, t \in [0, a)\,.$$

If Φ is causal, then for each $\tau \in [0, a)$ we have

$$(\tilde{\Phi}(u))(t) = (\Phi(\mathbf{Q}_\tau u))(t)\,, \quad \forall\, t \in [0, \tau]\,.$$

In the following, we shall use the same symbol Φ to denote the original operator acting on $C(\mathbb{R}_+, \mathbb{R})$ and the associated operator $\tilde{\Phi}$ acting on $C([0,a), \mathbb{R})$.

We introduce two assumptions on the nonlinearity $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$:

(A1) there exists $\lambda > 0$ such that for all $\alpha \in \mathbb{R}_+$ and all $w \in C([0,\alpha], \mathbb{R})$, there exist numbers $\delta_1, \delta_2 > 0$ such that for all $u, v \in C(w; \delta_1, \delta_2)$

$$\sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(u))(t) - (\Phi(v))(t)| \le \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |u(t) - v(t)|\,;$$

(A2) for all $a > 0$ and all $u \in C([0,a), \mathbb{R})$, there exist $\alpha, \beta > 0$ such that

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \le \alpha + \beta \sup_{t\in[0,\tau]} |u(t)|\,, \quad \forall\, \tau \in [0,a)\,.$$

In the following, Proposition 3.2.1 will be used to prove a global existence result for a general closed-loop system which encompasses the systems considered in later chapters. For $(A, B, C, D) \in \mathfrak{L}$ we consider

$$\dot{x}(t) = Ax(t) + B(\Phi(u))(t)\,, \quad x(0) = x_0 \in X\,, \tag{3.14a}$$
$$\dot{u}(t) = \kappa(t)\theta(t)[r - C_L x(t) - D(\Phi(u))(t)]\,, \quad u(0) = u_0 \in \mathbb{R}\,, \tag{3.14b}$$
$$\dot{\theta}(t) = h(\theta(t))|r - C_L x(t) - D(\Phi(u))(t)|\,, \quad \theta(0) = \theta_0 \in \mathbb{R}\,, \tag{3.14c}$$

where $\kappa \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

For $a \in (0, \infty]$, a continuous function

$$[0,a) \to X \times \mathbb{R} \times \mathbb{R}, \quad t \mapsto (x(t), u(t), \theta(t))$$

is a *solution* of (3.14) if $(x(\cdot), u(\cdot), \theta(\cdot))$ is absolutely continuous as a $(X_{-1}\times\mathbb{R}\times\mathbb{R})$-valued function, $x(t) \in \mathrm{dom}(C_L)$ for almost all $t \in [0,a)$, $(x(0), u(0), \theta(0)) = (x_0, u_0, \theta_0)$ and the differential equations in (3.14) are satisfied almost everywhere on $[0,a)$, where the derivative in (3.14a) should be interpreted in the space X_{-1}.†

On noting that $C_L x(t) + D(\Phi(u))(t) = (\mathbf{\Psi}_\infty x_0)(t) + (\mathbf{F}_\infty \Phi(u))(t)$, the variable $x(t)$ can be eliminated from (3.14b) and (3.14c) to obtain

$$\dot{u}(t) = \kappa(t)\theta(t)[r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \Phi(u))(t)]\,, \quad u(0) = u_0\,, \tag{3.15a}$$
$$\dot{\theta}(t) = h(\theta(t))|r - (\mathbf{\Psi}_\infty x_0)(t) - (\mathbf{F}_\infty \Phi(u))(t)|\,, \quad \theta(0) = \theta_0\,. \tag{3.15b}$$

In order to proceed we require the following lemma.

† Being a Hilbert space, $X_{-1} \times \mathbb{R} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [2], Theorem 3.1, p. 10.

Lemma 3.2.3 *Let $\Phi : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ be causal and satisfy (A1) and let $(A,B,C,D) \in \mathfrak{L}$. For all $\alpha \geq 0$ and all $w \in C([0,\alpha],\mathbb{R})$, there exist $\delta_1, \delta_2, \gamma_1, \gamma_2 > 0$ such that for all $\varepsilon \in [0,\delta_2]$ and $u, v \in C(w;\delta_1,\delta_2)$*

$$\int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty\Phi(u))(\tau) - (\mathbf{F}_\infty\Phi(v))(\tau)|\, d\tau \leq \varepsilon\gamma_1 \sup_{\alpha\leq\tau\leq\alpha+\varepsilon} |u(\tau) - v(\tau)|, \tag{3.16}$$

$$\int_\alpha^{\alpha+\varepsilon} |(\mathbf{F}_\infty\Phi(u))(\tau)|\, d\tau \leq \varepsilon\gamma_1\delta_1 + \sqrt{\varepsilon}\gamma_2. \tag{3.17}$$

Proof: Let $\alpha \geq 0$ and $w \in C([0,\alpha],\mathbb{R})$. Then by (A1) and causality of Φ, there exist numbers $\delta_1, \delta_2 > 0$ such that for all $\varepsilon \in [0,\delta_2]$ and all $u, v \in C(w;\delta_1,\delta_2)$

$$\begin{aligned}
\sup_{t\in[\alpha,\alpha+\varepsilon]} |(\Phi(u))(t) - (\Phi(v))(t)| &\leq \sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(\mathbf{Q}_{\alpha+\varepsilon}\, u))(t) - (\Phi(\mathbf{Q}_{\alpha+\varepsilon}\, v))(t)| \\
&\leq \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |(\mathbf{Q}_{\alpha+\varepsilon}\, u)(t) - (\mathbf{Q}_{\alpha+\varepsilon}\, v)(t)| \\
&= \lambda \sup_{t\in[\alpha,\alpha+\varepsilon]} |u(t) - v(t)|.
\end{aligned}$$

Hence using the causality of $\mathbf{F}_\infty$ and Φ, the boundedness of $\mathbf{F}_\infty$ as an operator from $L^2(\mathbb{R}_+,\mathbb{R})$ into $L^2(\mathbb{R}_+,\mathbb{R})$ and Hölder's inequality, we conclude that there exists $\gamma_1 > 0$ such that for all $\varepsilon \in [0,\delta_2]$ and all $u, v \in C(w;\delta_1,\delta_2)$

$$\begin{aligned}
\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\Phi(u) - \mathbf{F}_\infty\Phi(v)| &\leq \sqrt{\varepsilon}\left(\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\Phi(u) - \mathbf{F}_\infty\Phi(v)|^2\right)^{1/2} \\
&\leq \sqrt{\varepsilon}\|\mathbf{F}_\infty\|\left(\int_\alpha^{\alpha+\varepsilon} |\Phi(u) - \Phi(v)|^2\right)^{1/2} \\
&\leq \varepsilon\lambda\|\mathbf{F}_\infty\| \sup_{t\in[\alpha,\alpha+\varepsilon]} |u(t) - v(t)|.
\end{aligned}$$

which is (3.16) with $\gamma_1 := \lambda\|\mathbf{F}_\infty\|$. Moreover, an application of (3.16) for $v = \mathbf{Q}_\alpha u$ and Hölder's inequality show that for all $\varepsilon \in [0,\delta_2]$ and all $u \in C(w;\delta_1,\delta_2)$

$$\begin{aligned}
\int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\Phi(u)| &\leq \int_\alpha^{\alpha+\varepsilon} |\mathbf{F}_\infty\Phi(\mathbf{Q}_\alpha u)| + \varepsilon\gamma_1 \sup_{t\in[\alpha,\alpha+\varepsilon]} |u - v(\alpha)| \\
&\leq \sqrt{\varepsilon}\left(\int_\alpha^{\alpha+\delta_2} |\mathbf{F}_\infty\Phi(\mathbf{Q}_\alpha\, u)|^2\right)^{1/2} + \varepsilon\gamma_1\delta_1,
\end{aligned}$$

which yields (3.17) with $\gamma_2 := (\int_\alpha^{\alpha+\delta_2} |\mathbf{F}_\infty\Phi(\mathbf{Q}_\alpha\, u)|^2)^{1/2}$. □

The following corollary is the main result of the chapter.

Corollary 3.2.4 *Let $\Phi : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ be causal and satisfy (A1) and (A2). Let $(A,B,C,D) \in \mathfrak{L}$, $r \in \mathbb{R}$, $\kappa \in L^\infty(\mathbb{R}_+,\mathbb{R})$ and let $h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. If $h(\theta) \leq 0$ for all $\theta \in \mathbb{R}$ and $h(0) = 0$, then for all $(x_0, u_0, \theta_0) \in$*

$X \times \mathbb{R} \times (0, \infty)$, *the initial-value problem given by (3.14) has a unique solution defined on* $\mathbb{R}_+$.

Proof: Let $(x_0, u_0, \theta_0) \in X \times \mathbb{R} \times (0, \infty)$. It is clear that the map $V : C(\mathbb{R}_+, \mathbb{R}^2) \to L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^2)$ given by

$$V\begin{pmatrix} u \\ \theta \end{pmatrix}(t) = \begin{pmatrix} \kappa(t)\theta(t)[r - (\Psi_\infty x_0)(t) - (\mathbf{F}_\infty \Phi(u))(t)] \\ h(\theta(t))|r - (\Psi_\infty x_0)(t) - (\mathbf{F}_\infty \Phi(u))(t)| \end{pmatrix} \tag{3.18}$$

is causal, and it follows from Lemma 3.2.3 via a routine argument (shown in Appendix 2) that V is also weakly locally Lipschitz. Note that (3.15) is of the form (3.10) with V given by (3.18).
We proceed in three steps.

STEP 1. Existence and uniqueness on a small interval.

An application of Proposition 3.2.1 with $\alpha = 0$ shows that there exists an $\varepsilon > 0$ such that (3.10) has unique solution on the interval $[0, \varepsilon)$.

STEP 2. Extended uniqueness.

Let $v : [0, \beta_1) \to \mathbb{R}^2$ and $w : [0, \beta_2) \to \mathbb{R}^2$, $\beta_1, \beta_2 > 0$, be solutions of (3.10) (existence of v and w is assured by Step 1).

We claim that $v(t) = w(t)$ for all $t \in [0, \beta)$, where $\beta = \min\{\beta_1, \beta_2\}$. Seeking a contradiction, suppose that there exists $t \in (0, \beta)$ such that $v(t) \neq w(t)$. Defining

$$a^* = \inf\{t \in (0, \beta) \,|\, v(t) \neq w(t)\},$$

it follows that $a^* > 0$ (by Step 1), $a^* < \beta$ (by supposition) and $v(a^*) = w(a^*)$ (by continuity of v and w). Clearly, the initial-value problem

$$\dot{z}(t) = (Vz)(t), \quad t \geq a^*; \qquad z(t) = v(t), \quad t \in [0, a^*]$$

is solved by v and w on $[0, \beta)$. This implies, by Proposition 3.2.1 (with $\alpha = a^*$), that there exists an $\varepsilon > 0$ such that $v(t) = w(t)$ for all $t \in [0, a^* + \varepsilon)$, which contradicts the definition of a^*.

STEP 3. Global existence.

Let $I \subset \mathbb{R}_+$ be the set of all $\tau > 0$ such that there exists a solution (u^τ, θ^τ) of (3.15) on the interval $[0, \tau)$. Set $t^* := \sup I$ and define a function $(u, \theta) : [0, t^*) \to \mathbb{R}^2$ by setting

$$(u, \theta)(t) = (u^\tau, \theta^\tau)(t), \quad \text{for } t \in [0, \tau), \text{ where } \tau \in I.$$

By Step 2 the function (u, θ) is well-defined (i.e. the definition of $(u, \theta)(t)$ for a particular value $t \in [0, t^*)$ does not depend on the choice of $\tau \in I \cap (t, \infty)$) and (u, θ) is the unique solution of (3.15) on the interval $[0, t^*)$. We claim that $t^* = \infty$. Seeking a contradiction, assume that $t^* < \infty$. We first show that θ is bounded on $[0, t^*)$. Note that since $h \leq 0$, $\theta(\cdot)$ is non-increasing and combining this with the assumption that $\theta_0 > 0$, we see that boundeness of $\theta(\cdot)$ follows if we can show that $\theta(t) > 0$ for all $t \in [0, t^*)$. Seeking a contradiction, suppose that there exists a $\tau \in (0, t^*)$ such that $\theta(\tau) = 0$. Consider the following initial-value problem on $[0, t^*)$

$$\dot{\zeta}(t) = h(\zeta(t))|e(t)|, \quad \zeta(\tau) = 0, \tag{3.19}$$

where $e(t) = r - (\Psi_\infty x_0)(t) - (\mathbf{F}_\infty \Phi(u))(t)$. Then $\theta(\cdot)$ is a solution of (3.19). Since $h(0) = 0$, the function $\zeta \equiv 0$ is also a solution of (3.19). By uniqueness it follows that $\theta \equiv 0$, which is in contradiction to $\theta_0 > 0$. Therefore the function $\theta(\cdot)$ is bounded on $[0, t^*)$ and hence there exists a constant $\gamma > 0$ such that

$$|\kappa(t)\theta(t)| \leq \gamma, \quad \forall t \in [0, t^*).$$

Multiplying (3.15a) by u and estimating we obtain that, for all $t \in [0, t^*)$,

$$u(t)\dot{u}(t) \leq \gamma[r^2 + (\Psi_\infty x_0)^2(t) + u^2(t) + |(\mathbf{F}_\infty \Phi(u))(t)u(t)|]. \tag{3.20}$$

Integration yields

$$\begin{aligned} \frac{1}{2}u^2(t) \;&\leq\; \frac{1}{2}u^2(0) + \gamma\left(\int_0^t (r^2 + (\Psi_\infty x_0)^2)\right. \\ &\quad \left. + \int_0^t u^2 + \int_0^t |\mathbf{F}_\infty \Phi(u)||u|\right), \quad \forall t \in [0, t^*). \end{aligned} \tag{3.21}$$

For $v \in C([0, t^*), \mathbb{R})$ and $t \in [0, t^*)$, we define

$$\sigma_t(v) = \sup_{\tau \in [0,t]} |v(\tau)|.$$

Using (3.21), the boundedness of $\mathbf{F}_\infty$ as an operator from $L^2(\mathbb{R}_+, \mathbb{R})$ into $L^2(\mathbb{R}_+, \mathbb{R})$ and applying Hölder's inequality, shows that there exist $\gamma_1, \gamma_2 > 0$ such that for all $t \in [0, t^*)$,

$$\frac{1}{2}\sigma_t(u^2) \leq \frac{1}{2}u^2(0) + \gamma_1 + \gamma \int_0^t u^2 + \gamma_2 \left(\int_0^t \Phi(u)^2\right)^{1/2} \left(\int_0^t u^2\right)^{1/2}.$$

Denoting the map

$$C([0, t^*), \mathbb{R}) \to \mathbb{R}_+, \quad v \mapsto [\sigma_t(v)]^2$$

by σ_t^2, we see that there exist suitable constants $\gamma_3, \gamma_4, \gamma_5 > 0$ such that for all $t \in [0, t^*)$

$$\sigma_t^2(u) \le \gamma_3 + \gamma_4 \int_0^t \sigma_\tau^2(u)\, d\tau + \gamma_5 \left(\int_0^t \sigma_\tau^2(\Phi(u))\, d\tau \right)^{1/2} \left(\int_0^t \sigma_\tau^2(u)\, d\tau \right)^{1/2} .$$

Using asumption (A2), we may conclude that there exist numbers $\alpha, \beta > 0$ such that for all $t \in [0, t^*)$

$$\sigma_t^2(u) \le \gamma_3 + \gamma_4 \int_0^t \sigma_\tau^2(u)\, d\tau + \gamma_5 \left(\int_0^t [\alpha + \beta\sigma_\tau(u)]^2\, d\tau \right)^{1/2} \left(\int_0^t \sigma_\tau^2(u)\, d\tau \right)^{1/2} .$$

From this we obtain that there exist numbers $\gamma_6, \gamma_7 > 0$ such that

$$\sigma_t^2(u) \le \gamma_6 + \gamma_7 \int_0^t \sigma_\tau^2(u)\, d\tau\,, \quad \forall\, t \in [0, t^*)\,,$$

and an application of Gronwall's lemma then shows that

$$\sigma_t^2(u) \le \gamma_6 e^{\gamma_7 t}\,, \quad \forall\, t \in [0, t^*)\,.$$

Since, by assumption, $t^* < \infty$, it follows that u is bounded on $[0, t^*)$. Recall that κ and θ are also bounded on $[0, t^*)$ and thus by the local Lipschitz continuity of h, so is $h \circ \theta$. By (A2), it follows from the boundedness of u on $[0, t^*)$ that $\Phi(u)$ is bounded on $[0, t^*)$ and therefore $\Phi(u) \in L^2([0, t^*), \mathbb{R})$. Using the boundedness of $\mathbf{F}_\infty$ as an operator from $L^2(\mathbb{R}_+, \mathbb{R})$ into $L^2(\mathbb{R}_+, \mathbb{R})$, we obtain that $\mathbf{F}_\infty \Phi(u) \in L^2([0, t^*), \mathbb{R}) \subset L^1([0, t^*), \mathbb{R})$. Therefore, the right-hand sides of (3.15a) and (3.15b) are in $L^1([0, t^*), \mathbb{R})$, implying that $\lim_{t \uparrow t^*} u(t)$ and $\lim_{t \uparrow t^*} \theta(t)$ exist and are finite. Setting $u(t^*) = \lim_{t \uparrow t^*} u(t)$ and $\theta(t^*) = \lim_{t \uparrow t^*} \theta(t)$, makes u and θ into continuous functions on $[0, t^*]$. By Proposition 3.2.1, the initial-value problem

$$\dot{z}(t) = (Vz)(t)\,, \;\; t \ge t^*\,; \quad z(t) = (u, \theta)(t)\,, \;\; t \in [0, t^*]\,,$$

has a unique solution (u^*, θ^*) on $[0, t^* + \varepsilon)$ for some $\varepsilon > 0$. By causality of V, the function (u^*, θ^*) is a solution of (3.15) on $[0, t^* + \varepsilon)$, and so (u^*, θ^*) is a proper right continuation of (u, θ). But this means that $t^* + \varepsilon \in I$, which is in contradiction to the definition of t^*.

Finally, let $(u, \theta) : \mathbb{R}_+ \to \mathbb{R}^2$ be the unique solution of (3.15) and *define* $x(\cdot)$ to be the unique solution of

$$\dot{x} = Ax + B\Phi(u)\,, \quad x(0) = x_0\,.$$

Then $(x(\cdot), u(\cdot), \theta(\cdot))$ is the unique solution of (3.14) defined on $\mathbb{R}_+$. □

3.3 Notes and references

The existence and uniqueness results of Section 3.2, whilst new in this generality, are proved in a similar way to the less general existence and uniqueness results of [24] (see Appendix in [24]). In particular, we have a slightly weaker Lipschitz assumption on V in (3.10). This implies that Lemma 3.2.3 is slightly stronger than the similar lemma in [24] (see Lemma 5.3 in [24]). We essentially have proved the same results as in the Appendix of [24] but with weaker assumptions.

Chapter 4

Hysteresis operators

4.1 Continuous-time hysteresis operators

In this section we present basic background material on hysteresis operators which is needed for the subsequent developments in this chapter.

We call a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ a *time transformation* if f is continuous, non-decreasing and satisfies $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$, in other words $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a time transformation if and only if f is continuous, non-decreasing and surjective. We denote the set of all time transformations $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mathfrak{T}$.

In the following let $\mathfrak{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathfrak{F} \neq \emptyset$. We introduce the following two assumptions on $\mathfrak{F}$:

(F1) $u \circ f \in \mathfrak{F}$ for all $u \in \mathfrak{F}$ and all $f \in \mathfrak{T}$;

(F2) $\mathbf{Q}_t(\mathfrak{F}) \subset \mathfrak{F}$ for all $t \in \mathbb{R}_+$.

An operator $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called *rate independent* if $\mathfrak{F}$ satisfies (F1) and

$$(\Phi(u \circ f))(t) = (\Phi(u))(f(t)), \quad \forall\, u \in \mathfrak{F}, \quad \forall\, f \in \mathfrak{T}, \quad \forall\, t \in \mathbb{R}_+ .$$

A functional $\varphi : \mathfrak{F} \to \mathbb{R}$ is called *rate independent* if $\mathfrak{F}$ satisfies (F1) and

$$\varphi(u \circ f) = \varphi(u), \quad \forall\, u \in \mathfrak{F}, \quad \forall\, f \in \mathfrak{T}.$$

Definition 4.1.1 Let $\mathfrak{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathfrak{F} \neq \emptyset$. An operator $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$ is called a *hysteresis operator* if $\mathfrak{F}$ satisfies (F1) and Φ is causal and rate independent. $\diamond$

For $\mathfrak{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathfrak{F} \neq \emptyset$, let $\mathfrak{F}^{\mathrm{uc}}$ denote the set of all ultimately constant $u \in \mathfrak{F}$, i.e.

$$\mathfrak{F}^{\mathrm{uc}} = \{u \in \mathfrak{F} \,|\, u \text{ is ultimately constant}\}.$$

Clearly, if $\mathfrak{F}$ satisfies (F2), then $\mathfrak{F}^{\mathrm{uc}} \neq \emptyset$. Moreover, if $\mathfrak{F}$ satisfies (F1), then so does $\mathfrak{F}^{\mathrm{uc}}$.

Theorem 4.1.2 *Let $\mathfrak{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathfrak{F} \neq \emptyset$ and assume that (F1) and (F2) hold. If $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then the following statements hold:*

(1) for all $u \in \mathfrak{F}$ and all $\tau \in \mathbb{R}_+$

$$(\Phi(\mathbf{Q}_\tau u))(t) = (\Phi(u))(\tau), \quad \forall\, t \geq \tau;$$

(2) the functional

$$\varphi : \mathfrak{F}^{\mathrm{uc}} \to \mathbb{R}, \quad u \mapsto \lim_{t\to\infty} (\Phi(u))(t), \tag{4.1}$$

is rate independent and satisfies

$$(\Phi(u))(t) = \varphi(\mathbf{Q}_t u), \quad \forall\, u \in \mathfrak{F}, \quad \forall\, t \in \mathbb{R}_+. \tag{4.2}$$

Conversely, if $\varphi : \mathfrak{F}^{\mathrm{uc}} \to \mathbb{R}$ is a rate independent functional, then $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$ given by (4.2) is a hysteresis operator and satisfies

$$\lim_{t\to\infty} (\Phi(u))(t) = \varphi(u), \quad \forall\, u \in \mathfrak{F}^{\mathrm{uc}}. \tag{4.3}$$

For a hysteresis operator $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$, we call the rate independent functional $\varphi : \mathfrak{F}^{\mathrm{uc}} \to \mathbb{R}$ defined by (4.1) the *representing functional* of Φ.

Proof of Theorem 4.1.2: Assume that $\Phi : \mathfrak{F} \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator. To prove statement (1), let $u \in \mathfrak{F}$, $\tau \in \mathbb{R}_+$ and $s > \tau$. We define a time transformation $f \in \mathfrak{T}$ by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq \tau, \\ \tau & \text{for } \tau < t \leq s, \\ t + \tau - s & \text{for } t > s. \end{cases}$$

Then, using the causality and rate independence of Φ, we have for $t \in [\tau, s]$

$$(\Phi(\mathbf{Q}_\tau u))(t) = (\Phi(u \circ f))(t) = (\Phi(u))(f(t)) = (\Phi(u))(\tau).$$

Since $s > \tau$ was arbitrary, this yields statement (1). To prove statement (2), we first note that the limit in (4.1) exists since for ultimately constant u, $\Phi(u)$ is ultimately constant by statement (1). Using the rate independence of Φ, we see that for all $u \in \mathcal{F}^{\mathrm{uc}}$ and all $f \in \mathcal{T}$

$$\varphi(u \circ f) = \lim_{t\to\infty} (\Phi(u \circ f))(t) = \lim_{t\to\infty} (\Phi(u))(f(t)) = \lim_{t\to\infty} (\Phi(u))(t) = \varphi(u)\,,$$

showing that φ is rate independent. Using statement (1), we obtain for all $u \in \mathcal{F}$ and all $t \in \mathbb{R}_+$

$$(\Phi(u))(t) = (\Phi(\mathbf{Q}_t\, u))(t) = \lim_{s\to\infty} (\Phi(\mathbf{Q}_t\, u))(s) = \varphi(\mathbf{Q}_t\, u)\,,$$

which is (4.2).

Conversely, assume that $\varphi : \mathcal{F}^{\mathrm{uc}} \to \mathbb{R}$ is rate independent and define $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ by (4.2). Then, trivially, Φ is causal. Moreover, for all $u \in \mathcal{F}$, $f \in \mathcal{T}$ and $t \in \mathbb{R}_+$

$$(\Phi(u \circ f))(t) = \varphi(\mathbf{Q}_t\,(u \circ f)) = \varphi((\mathbf{Q}_{f(t)}\, u) \circ f) = \varphi(\mathbf{Q}_{f(t)}\, u) = (\Phi(u))(f(t))\,,$$

thus Φ is rate independent. Finally, let $u \in \mathcal{F}^{\mathrm{uc}}$, then

$$\lim_{t\to\infty} (\Phi(u))(t) = \lim_{t\to\infty} \varphi(\mathbf{Q}_t u) = \varphi(u)\,,$$

which is (4.3). $\square$

Let $\mathcal{S}^r$ denote the set of all right-continuous step functions $u : \mathbb{R}_+ \to \mathbb{R}$, that is there exists a sequence $0 = t_0 < t_1 < t_2 < \ldots$ such that $\lim_{n\to\infty} t_n = \infty$ and u is constant on each of the intervals $[t_i, t_{i+1})$. For $\tau > 0$ define $\mathcal{S}^r_\tau \subset \mathcal{S}^r$ to be the set of all right-continuous step functions $u : \mathbb{R}_+ \to \mathbb{R}$ of step length τ, i.e. u is constant on each interval $[i\tau, (i+1)\tau)$. We note that whilst $\mathcal{S}^r$ satisfies (F1) and (F2), $\mathcal{S}^r_\tau$ satisfies (F2), but not (F1). The following corollary is an immediate consequence of Theorem 4.1.2, statement (1).

Corollary 4.1.3 *Let $\mathcal{F} \subset F(\mathbb{R}_+, \mathbb{R})$, $\mathcal{F} \neq \emptyset$ and assume that (F1) and (F2) hold. Let $\Phi : \mathcal{F} \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Then*

$$\Phi(\mathcal{F}^{\mathrm{uc}}) \subset \mathcal{F}^{\mathrm{uc}}\,, \quad \Phi(\mathcal{F} \cap \mathcal{S}^r) \subset \mathcal{S}^r\,, \quad \Phi(\mathcal{F} \cap \mathcal{S}^r_\tau) \subset \mathcal{S}^r_\tau\,.$$

For any $u \in F(\mathbb{R}_+, \mathbb{R})$ and any $t \in \mathbb{R}_+$, we define

$$M(u,t) := \{\tau \in (t, \infty) \,|\, u \text{ is monotone on } (t, \tau)\}\,.$$

If u is piecewise monotone, then $M(u,t) \neq \emptyset$ for all $t \in \mathbb{R}_+$, and the *standard monotonicity partition* $t_0 < t_1 < t_2 < \ldots$ of u is defined recursively by setting

$t_0 = 0$ and $t_{i+1} = \sup M(u, t_i)$ for all $i \in \mathbb{Z}_+$ such that $M(u, t_i)$ is bounded. If u is piecewise monotone and ultimately constant, then the standard monotonicity partition of u is finite.

We define $C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ to be the set of all ultimately constant $u \in C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. We note that $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and $C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ both satisfy (F1) and (F2). Let $F^{\rm uc}(\mathbb{Z}_+, \mathbb{R})$ denote the space of ultimately constant $u : \mathbb{Z}_+ \to \mathbb{R}$. We define the *restriction operator* $R : C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F^{\rm uc}(\mathbb{Z}_+, \mathbb{R})$ by

$$(R(u))(k) = \begin{cases} u(t_k) & \text{for } k \in [0, m] \cap \mathbb{Z}_+\,, \\ \lim_{t\to\infty} u(t) & \text{for } k \in \mathbb{Z}_+ \setminus [0, m]\,, \end{cases}$$

where $0 = t_0 < t_1 < \ldots < t_m$ is the standard monotonicity partition of u.

The following lemma will be an important tool in the next section.

Lemma 4.1.4 *Let $u, v \in C^{\rm uc}_{\rm pm}(\mathbb{R}_+, \mathbb{R})$. Then $R(u) = R(v)$ if and only if there exist $f, g \in \mathcal{T}$ such that $u \circ f = v \circ g$.*

The above lemma is essentially due to Brokate and Sprekels [4] (see lemma 2.2.4 in [4]). Since only a sketch of the proof is given in [4], we have included a complete proof in the Appendices (see Appendix 3).

As an immediate consequence of Lemma 4.1.4 and Theorem 4.1.2, statement (2), we obtain the following corollary.

Corollary 4.1.5 *Let $\Phi : C_{\rm pm}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator, $u, v \in C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. Then*

$$R(\mathbf{Q}_t\, u) = R(\mathbf{Q}_t\, v) \quad \Longrightarrow \quad (\Phi(u))(t) = (\Phi(v))(t)\,.$$

The above corollary says that the output $(\Phi(u))(t)$ at time $t \in \mathbb{R}_+$ of a hysteresis operator Φ corresponding to a continuous piecewise monotone input u is determined completely by the local extrema of u restricted to the time interval $[0, t]$.

4.2 Extending Lipschitz continuous hysteresis operators defined on $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ to $C(\mathbb{R}_+, \mathbb{R})$

The following lemma shows that the range of a Lipschitz continuous hysteresis operator defined on $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ or $C(\mathbb{R}_+, \mathbb{R})$ is contained in $C(\mathbb{R}_+, \mathbb{R})$.

Lemma 4.2.1 *Let $\mathcal{C} = C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ and let $\Phi : \mathcal{C} \to F(\mathbb{R}_+, \mathbb{R})$ be a Lipschitz continuous hysteresis operator. Then $\Phi(\mathcal{C}) \subset C(\mathbb{R}_+, \mathbb{R})$.*

Proof: Let $u \in \mathcal{C}$, $\varepsilon > 0$, $t \in \mathbb{R}_+$ and let $\lambda > 0$ be a Lipschitz constant of Φ. By the continuity of u, there exists $\delta > 0$ such that for all $s \in \mathbb{R}_+$

$$|s - t| < \delta \quad \Longrightarrow \quad |u(s) - u(t)| < \varepsilon/(2\lambda)\,. \tag{4.4}$$

Let $\tau \in \mathbb{R}_+$ be such that $|\tau - t| < \delta$. If $\tau > t$, then using Theorem 4.1.2, statement (1) and (4.4)

$$\begin{aligned}
|(\Phi(u))(\tau) - (\Phi(u))(t)| &= |(\Phi(\mathbf{Q}_\tau\, u))(\tau) - (\Phi(\mathbf{Q}_t\, u))(\tau)| \\
&\leq \lambda \sup_{s \in \mathbb{R}_+} |(\mathbf{Q}_\tau\, u)(s) - (\mathbf{Q}_t\, u)(s)| \\
&= \lambda \sup_{s \in [t,\tau]} |u(s) - u(t)| < \varepsilon\,.
\end{aligned}$$

If $\tau < t$, then again using Theorem 4.1.2, statement (1) and (4.4)

$$\begin{aligned}
|(\Phi(u))(\tau) - (\Phi(u))(t)| &= |(\Phi(\mathbf{Q}_\tau\, u))(t) - (\Phi(\mathbf{Q}_t\, u))(t)| \\
&\leq \lambda \sup_{s \in \mathbb{R}_+} |(\mathbf{Q}_\tau\, u)(s) - (\mathbf{Q}_t\, u)(s)| \\
&= \lambda \sup_{s \in [\tau,t]} |u(\tau) - u(s)| \\
&\leq \lambda |u(\tau) - u(t)| + \lambda \sup_{s \in [\tau,t]} |u(t) - u(s)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon\,.
\end{aligned}$$

□

The following two propositions are the main results of this section.

Proposition 4.2.2 *Let $\Phi : C_{\text{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be a Lipschitz continuous hysteresis operator with Lipschitz constant $\lambda > 0$. Then there exists a unique Lipschitz continuous extension $\Phi_e : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ with Lipschitz constant λ. Moreover, Φ_e is a hysteresis operator.*

Proof: By Lemma 2.1.4 we know that there exists a unique Lipschitz continuous extension $\Phi_e : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, with Lipschitz constant λ. Moreover, the causality of Φ_e follows easily from the causality of Φ. To show rate independence, let $u \in C(\mathbb{R}_+, \mathbb{R})$ and $f \in \mathcal{T}$. Choose $(u_n) \subset C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $u_n \overset{\text{uc}}{\to} u$. Then $\Phi(u_n) \overset{\text{uc}}{\to} \Phi_e(u)$ as $n \to \infty$,

$$u_n \circ f \overset{\text{uc}}{\to} u \circ f \quad \text{as } n \to \infty\,, \tag{4.5}$$

and

$$\Phi(u_n) \circ f \stackrel{\text{uc}}{\to} \Phi_e(u) \circ f \quad \text{as } n \to \infty . \tag{4.6}$$

Now (4.5) implies that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}_+} |(\Phi(u_n \circ f))(t) - (\Phi_e(u \circ f))(t)| = 0 . \tag{4.7}$$

By the rate independence of Φ, $\Phi(u_n \circ f) = \Phi(u_n) \circ f$ for all $n \in \mathbb{Z}_+$ and therefore (4.6) and (4.7) imply that $\Phi_e(u \circ f) = \Phi_e(u) \circ f$. □

Proposition 4.2.3 *Let* $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ *or* $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ *and let* $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ *be a Lipschitz continuous hysteresis operator with Lipschitz constant* $\lambda > 0$. *Then* $\Phi(AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}) \subset AC(\mathbb{R}_+, \mathbb{R})$.

Proof: Let $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$, $\varepsilon > 0$ and $b > a \geq 0$. Then there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |u(b_k) - u(a_k)| \leq \frac{\varepsilon}{l} ,$$

for every finite family of pairwise disjoint subintervals $(a_k, b_k) \subset [a, b]$ of total length

$$\sum_{k=1}^{n} (b_k - a_k) \leq \delta . \tag{4.8}$$

Since u is continuous, there exists $c_k \in [a_k, b_k]$ such that

$$|u(c_k) - u(a_k)| = \max_{t\in[a_k,b_k]} |u(t) - u(a_k)| .$$

Using the Lipschitz continuity of Φ and Theorem 4.1.2, statement (1), we obtain for any $\tau_1, \tau_2 \in \mathbb{R}_+$ with $\tau_2 \geq \tau_1$

$$\begin{aligned} |(\Phi(u))(\tau_2) - (\Phi(u))(\tau_1)| &= |(\Phi(\mathbf{Q}_{\tau_2} u))(\tau_2) - (\Phi(\mathbf{Q}_{\tau_1} u))(\tau_2)| \\ &\leq \lambda \sup_{t\in\mathbb{R}_+} |(\mathbf{Q}_{\tau_2} u)(t) - (\mathbf{Q}_{\tau_1} u)(t)| \\ &= \lambda \max_{t\in[\tau_1,\tau_2]} |u(t) - u(\tau_1)| . \end{aligned} \tag{4.9}$$

Now suppose that the family of intervals (a_k, b_k) satisfies (4.8). Then

$$\sum_{k=1}^{n} (c_k - a_k) \leq \delta ,$$

and so

$$\sum_{k=1}^{n} \max_{t\in[a_k,b_k]} |u(t) - u(a_k)| = \sum_{k=1}^{n} |u(c_k) - u(a_k)| \leq \frac{\varepsilon}{\lambda} . \tag{4.10}$$

Using (4.9) and (4.10), we may conclude

$$\sum_{k=1}^{n} |(\Phi(u))(b_k) - (\Phi(u))(a_k)| \le \lambda \sum_{k=1}^{n} \max_{t\in[a_k,b_k]} |u(t) - u(a_k)| \le \varepsilon\,,$$

showing that $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$. □

4.3 Examples of hysteresis operators

For $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ we define a set of partitions of $\mathbb{R}_+$ by

$$\begin{aligned} P_u \;:=\; \{(t_i) \subset \mathbb{R}_+ \mid t_0 = 0, (t_i) \text{ is strictly increasing, } \lim_{n\to\infty} t_n = \infty \\ \text{and } u \text{ is monotone on each of the intervals } [t_i, t_{i+1}]\}\,. \end{aligned}$$

We now introduce some well known operators and show that they are hysteresis operators.

Static nonlinearities

Although static nonlinearities do not describe hysteresis phenomena, we include them here because they form a special subclass of hysteresis operators.

For $\phi \in F(\mathbb{R}, \mathbb{R})$, define the corresponding static nonlinearity by

$$\mathcal{S}_\phi : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})\,, \quad u \mapsto \phi \circ u\,.$$

Trivially, $\mathcal{S}_\phi$ is a hysteresis operator.

Relay hysteresis

In *relay* (also called *passive* or *positive*) *hysteresis*, the relationship between input and output is determined by two threshold values $a_1 < a_2$ for the input. The output $v(t) = (\mathcal{R}_\xi(u))(t)$ moves, for a given continuous input $u(t)$, on one of two fixed curves $\rho_1 : [a_1, \infty) \to \mathbb{R}$ and $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ (see Figure 6), depending on which threshold, a_1 or a_2, was last attained.

More formally, let $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$ and let $\rho_1 : [a_1, \infty) \to \mathbb{R}$ and $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ be continuous. For $u \in C(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$ define

$$S(u,t) := u^{-1}(\{a_1, a_2\}) \cap [0,t]\,, \quad \tau(u,t) := \begin{cases} \max S(u,t) & \text{if } S(u,t) \ne \emptyset\,, \\ -1 & \text{if } S(u,t) = \emptyset\,. \end{cases} \tag{4.11}$$

Following Macki et al. [31], for each $\xi \in \mathbb{R}$, we define an operator $\mathcal{R}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(\mathcal{R}_\xi(u))(t) = \begin{cases} \rho_2(u(t)) & \text{if } u(t) \leq a_1, \\ \rho_1(u(t)) & \text{if } u(t) \geq a_2, \\ \rho_2(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) \neq -1,\ u(\tau(u,t)) = a_1, \\ \rho_1(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) \neq -1,\ u(\tau(u,t)) = a_2, \\ \rho_1(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) = -1,\ \xi > 0, \\ \rho_2(u(t)) & \text{if } u(t) \in (a_1, a_2),\ \tau(u,t) = -1,\ \xi \leq 0. \end{cases} \tag{4.12}$$

The number ξ plays the role of an "initial state" which determines the output value $(\mathcal{R}_\xi(u))(t)$ if $u(s) \in (a_1, a_2)$ for all $s \in [0, t]$. The operator $\mathcal{R}_\xi$ is called a *relay hysteresis operator* and is illustrated in Figure 6.

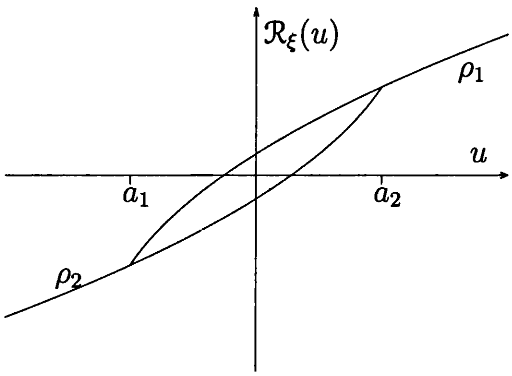

Figure 6: Relay hysteresis

To see that $\mathcal{R}_\xi$ is a hysteresis operator in the sense of Definition 4.1.1, note first that causality of $\mathcal{R}_\xi$ is immediate. To show rate independence of $\mathcal{R}_\xi$, let $u \in C(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$ and $f \in \mathcal{T}$. Then, clearly, $S(u \circ f, t) = \emptyset$ if and only if $S(u, f(t)) = \emptyset$; if $S(u \circ f, t) \neq \emptyset$, then it is clear that $f(\tau(u \circ f, t)) = \tau(u, f(t))$. Therefore we may conclude that $\mathcal{R}_\xi$ is rate independent.

Generalized backlash hysteresis

The *backlash* operator (also called *play* operator) has been discussed in a mathematically rigorous context in a number of references, see for example [3], [4], [16] and [39]. Intuitively, the backlash operator describes the input-output behaviour of a simple mechanical play between two mechanical elements I and II shown in Figure 7. The position of element I at time t is denoted by $u(t)$. The position $v(t)$ of the middle point of element II at time t will remain constant as long as $u(t)$ moves in the interior and it will change at the rate $\dot{v}=\dot{u}$ as long as $u(t)$

hits the boundary of element II with a velocity which is directed outwards. We first introduce the *generalized backlash operator* of which the (standard) backlash operator is an important example. Generalized backlash (also called *generalized play*) was introduced in [16].

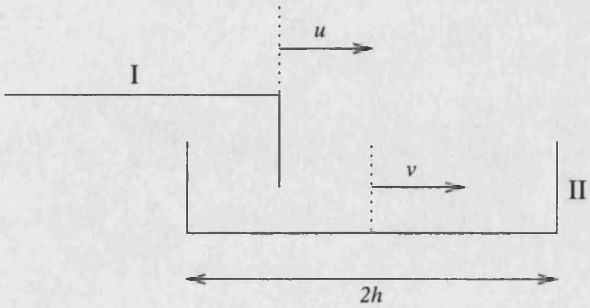

Figure 7: Schematic representation of backlash

Let $\beta_1, \beta_2 \in C(\mathbb{R}, \mathbb{R})$ be non-decreasing, globally Lipschitz with Lipschitz constant $\lambda > 0$, and such that $\operatorname{im} \beta_1 = \operatorname{im} \beta_2$ and $\beta_1(v) \le \beta_2(v)$ for all $v \in \mathbb{R}$. To give a formal definition of generalized backlash, define the function $b : \mathbb{R}^2 \to \mathbb{R}$ by

$$b(v, w) = \max\{\beta_1(v), \min\{\beta_2(v), w\}\} . \tag{4.13}$$

Note that

$$b(v, w) \in [\beta_1(v), \beta_2(v)] , \quad \forall (v, w) \in \mathbb{R}^2 . \tag{4.14}$$

The following "semigroup property" will prove useful when deriving properties of the generalized backlash operator.

Lemma 4.3.1 *Let $t_2 > t_1 \ge 0$, $u : [t_1, t_2] \to \mathbb{R}$ be monotone and $w \in [\beta_1(u(t_1)), \beta_2(u(t_1))]$. Then, for all $t, \tau \in [t_1, t_2]$ with $t \ge \tau$,*

$$b(u(t), w) = b(u(t), b(u(\tau), w)) .$$

Proof: Let $t_2 > t_1 \ge 0$, $u : [t_1, t_2] \to \mathbb{R}$ be monotone and $w \in [\beta_1(u(t_1)), \beta_2(u(t_1))]$. Fix $t, \tau \in [t_1, t_2]$ with $t \ge \tau$. We first note that $w = b(u(t_1), w)$ since $w \in [\beta_1(u(t_1)), \beta_2(u(t_1))]$. Without loss of generality we may assume that u is non-decreasing and so $w = b(u(t_1), w) \le b(u(\tau), w)$. If $w = b(u(\tau), w)$, then, trivially, $b(u(t), w) = b(u(t), b(u(\tau), w))$. If $w < b(u(\tau), w)$, then $b(u(\tau), w) = \beta_1(u(\tau))$ and thus $w < b(u(\tau), w) \le \beta_1(u(t))$, since β_1 and u are non-decreasing. Consequently,

$$b(u(t), w) = \beta_1(u(t)) = b(u(t), b(u(\tau), w)) .$$

□

For all $\xi \in \mathbb{R}$ we introduce an operator $\mathcal{B}_\xi$ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by defining recursively for every $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

$$(\mathcal{B}_\xi(u))(t) = \begin{cases} b(u(0), \xi) & \text{for } t = 0\,, \\ b(u(t), (\mathcal{B}_\xi(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i \in \mathbb{Z}_+\,, \end{cases} \tag{4.15}$$

where $(t_i) \in P_u$. Again, ξ plays the role of an "initial state".

Let $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We now show that the definition of $\mathcal{B}_\xi(u)$ is independent of the choice of partition. Let $(t_i), (\tau_i) \in P_u$. Without loss of generality we may assume that $\{\tau_i\} \subset \{t_i\}$. Define

$$(\mathcal{B}^1_\xi(u))(t) = \begin{cases} b(u(0), \xi) & \text{for } t = 0\,, \\ b(u(t), (\mathcal{B}^1_\xi(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i \in \mathbb{Z}_+\,, \end{cases}$$

and

$$(\mathcal{B}^2_\xi(u))(t) = \begin{cases} b(u(0), \xi) & \text{for } t = 0\,, \\ b(u(t), (\mathcal{B}^2_\xi(u))(\tau_i)) & \text{for } \tau_i < t \le \tau_{i+1},\ i \in \mathbb{Z}_+\,. \end{cases}$$

Obviously $(\mathcal{B}^1_\xi(u))(0) = (\mathcal{B}^2_\xi(u))(0)$. It is sufficient to show that if $(\mathcal{B}^1_\xi(u))(\tau_k) = (\mathcal{B}^2_\xi(u))(\tau_k)$, then $(\mathcal{B}^1_\xi(u))(t) = (\mathcal{B}^2_\xi(u))(t)$ for all $t \in (\tau_k, \tau_{k+1}]$. Let us suppose that for $k \in \mathbb{Z}_+$, $(\mathcal{B}^1_\xi(u))(\tau_k) = (\mathcal{B}^2_\xi(u))(\tau_k)$. Let $t \in (\tau_k, \tau_{k+1}]$ and choose $j \in \mathbb{Z}_+$ such that $t \in (t_j, t_{j+1}]$. Since $\{\tau_i\} \subset \{t_i\}$, $t_j \ge \tau_k$. If $t_j = \tau_k$ then, trivially, $(\mathcal{B}^1_\xi(u))(t) = (\mathcal{B}^2_\xi(u))(t)$. If $t_j > \tau_k$ then, since $(\mathcal{B}^1_\xi(u))(\tau) \in [\beta_1(u(\tau)), \beta_2(u(\tau))]$ for all $\tau \in \mathbb{R}_+$ (by (4.14)) and since there exists $i < j$ such that $t_i = \tau_k$, a repeated application of Lemma 4.3.1 gives

$$(\mathcal{B}^1_\xi(u))(t) = b(u(t), (\mathcal{B}^1_\xi(u))(t_j)) = b(u(t), (\mathcal{B}^1_\xi(u))(t_i)) = (\mathcal{B}^2_\xi(u))(t)\,.$$

The operator $\mathcal{B}_\xi$ is is called the *generalized backlash operator* and is illustrated in Figure 8.

Proposition 4.3.2 *Let $\xi \in \mathbb{R}$. Then*

(1) $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ *is a hysteresis operator;*

(2) $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ *is a Lipschitz continuous operator with Lipschitz constant* $l = \lambda$ *and* $\mathcal{B}_\xi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$*;*

(3) $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *extends to a Lipschitz continuous hysteresis operator* $\mathcal{B}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ *with Lipschitz constant* $l = \lambda$.

Proof: To prove that $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, we first note that $\mathcal{B}_\xi$ is causal. To show that $\mathcal{B}_\xi$ is rate independent, let $u \in$

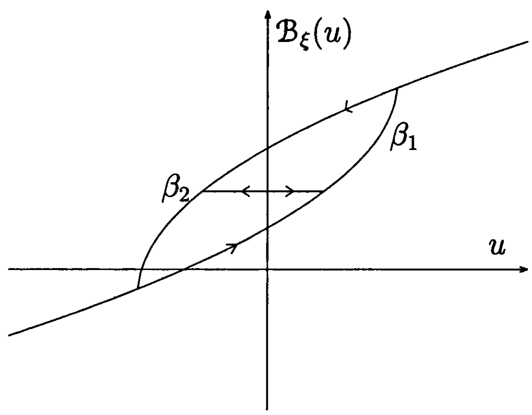

Figure 8: Generalized backlash hysteresis

$C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, $f \in \mathcal{T}$ and let $(t_i) \in P_{u\circ f}$ be such that $f(t_i) < f(t_{i+1})$ for all $i \in \mathbb{Z}_+$. Then $(f(t_i)) \in P_u$. Note that $(\mathcal{B}_\xi(u \circ f))(0) = (\mathcal{B}_\xi(u))(f(0))$ and suppose that for some $i \in \mathbb{Z}_+$, $(\mathcal{B}_\xi(u \circ f))(t_i) = (\mathcal{B}_\xi(u))(f(t_i))$. To prove rate independence, it is sufficient to show that $(\mathcal{B}_\xi(u \circ f))(t) = (\mathcal{B}_\xi(u))(f(t))$ for all $t \in (t_i, t_{i+1}]$. To this end let $t \in (t_i, t_{i+1}]$. Then

$$\begin{aligned}(\mathcal{B}_\xi(u \circ f))(t) &= b((u \circ f)(t), (\mathcal{B}_\xi(u \circ f))(t_i)) \\ &= b(u(f(t)), (\mathcal{B}_\xi(u))(f(t_i)) = (\mathcal{B}_\xi(u))(f(t)) . \end{aligned} \tag{4.16}$$

To show that $\mathcal{B}_\xi : C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F(\mathbb{R}_+,\mathbb{R})$ is a Lipschitz continuous operator with Lipschitz constant $l = \lambda$, note that for all $v_1, v_2, w_1, w_2 \in \mathbb{R}$

$$|\max\{v_1, v_2\} - \max\{w_1, w_2\}| \le \max\{|v_1 - w_1|, |v_2 - w_2|\} , \tag{4.17}$$

and

$$|\min\{v_1, v_2\} - \min\{w_1, w_2\}| \le \max\{|v_1 - w_1|, |v_2 - w_2|\} . \tag{4.18}$$

The above two inequalities imply that

$$|b(v_1, v_2) - b(w_1, w_2)| \le \max\{\lambda|v_1 - w_1|, |v_2 - w_2|\} . \tag{4.19}$$

Let $u, v \in C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and $(t_i) \in P_u \cap P_v$. For $t > 0$, let $j \in \mathbb{Z}_+$ be such that $t \in (t_j, t_{j+1}]$. Then by induction using (4.15) and (4.19)

$$\begin{aligned}|(\mathcal{B}_\xi(u))(t) - (\mathcal{B}_\xi(v))(t)| &= |b(u(t), (\mathcal{B}_\xi(u))(t_j)) - b(v(t), (\mathcal{B}_\xi(v))(t_j))| \\ &\le \max\{\lambda|u(t) - v(t)|, |(\mathcal{B}_\xi(u))(t_j) - (\mathcal{B}_\xi(v))(t_j)|\} \\ &\le \max\{\lambda|u(t) - v(t)|, \lambda \max_{0\le i\le j} |u(t_i) - v(t_i)|\}\end{aligned}$$

$$\leq \lambda \max_{0\leq\tau\leq t} |u(\tau) - v(\tau)|$$
$$\leq \lambda \sup_{\tau\in\mathbb{R}_+} |u(\tau) - v(\tau)| .$$

Thus $\mathcal{B}_\xi$ is Lipschitz continuous and by Lemma 4.2.1, $\mathcal{B}_\xi(C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Statement (3) follows from statements (1) and (2) combined with Proposition 4.2.2. □

Standard backlash hysteresis

Let $h \in \mathbb{R}_+$ and $\beta_1, \beta_2 : \mathbb{R} \to \mathbb{R}$ be given by $\beta_1(v) = v - h$ and $\beta_2(v) = v + h$. The function b defined by (4.13) then becomes

$$b(v, w) = \max\{v - h, \min\{v + h, w\}\} =: b_h(v, w) . \tag{4.20}$$

By replacing b by b_h on the right-hand side of (4.15), we obtain the (standard) backlash hysteresis operator $\mathcal{B}_{h,\xi} : C_{\text{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$. The *backlash operator* $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is illustrated in Figure 9.

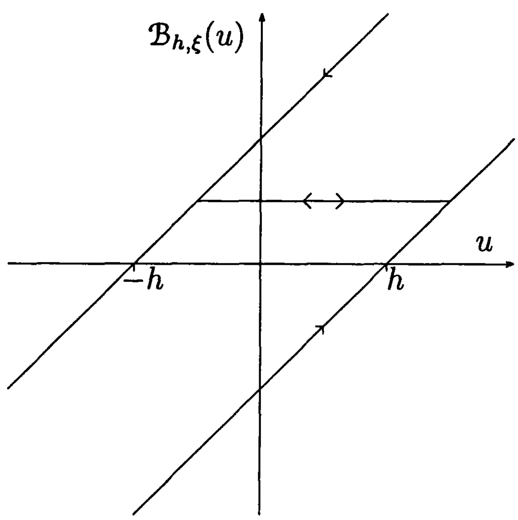

Figure 9: Backlash hysteresis

For future reference we state the following lemma which is an immediate consequence of (4.17) and (4.18).

Lemma 4.3.3 *For all $h_1, h_2 \in \mathbb{R}_+$ and all $v_1, v_2, w_1, w_2 \in \mathbb{R}$,*

$$|b_{h_1}(v_1, w_1) - b_{h_2}(v_2, w_2)| \leq \max\{|(v_1 - v_2) + (h_2 - h_1)|, |w_1 - w_2|\} .$$

Elastic-plastic hysteresis

The *elastic-plastic* operator (also called *stop* operator) models the stress-strain relationship in a one-dimensional elastic-plastic element. As long as the modulus of the stress v is smaller than the yield stress h, the strain u is related to v through the linear Hooke's Law. Once the stress exceeds the yield value it remains constant under further increasing of the strain; however, the elastic behaviour is instantly recovered when the strain is again decreased. As we shall see, elastic-plastic hysteresis is closely related to backlash hysteresis.

To give a formal definition of the elastic-plastic operator, define for each $h \in \mathbb{R}_+$ the function $e_h : \mathbb{R} \to \mathbb{R}$ by

$$e_h(v) = \min\{h, \max\{-h, v\}\}. \tag{4.21}$$

From (4.20) and (4.21), we see that for any $v, w \in \mathbb{R}$

$$\begin{aligned} v - b_h(v,w) &= v - \max\{v - h, \min\{v + h, w\}\} \\ &= \min\{h, \max\{-h, v - w\}\} = e_h(v - w). \end{aligned} \tag{4.22}$$

Following [4], for all $h \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$, we introduce an operator $\mathcal{E}_{h,\xi}$ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by defining recursively for every $u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

$$(\mathcal{E}_{h,\xi}(u))(t) = \begin{cases} e_h(u(0) - \xi) & \text{for } t = 0, \\ e_h(u(t) - u(t_i) + (\mathcal{E}_{h,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i \in \mathbb{Z}_+, \end{cases} \tag{4.23}$$

where $(t_i) \in P_u$. As with backlash we note that the definition is independent of the choice of partition. The operator $\mathcal{E}_{h,\xi}$ is called the *elastic-plastic operator* and is illustrated in Figure 10.

It follows immediately from (4.22) that

$$\mathcal{B}_{h,\xi}(u) + \mathcal{E}_{h,\xi}(u) = u, \quad \forall\, u \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}). \tag{4.24}$$

Proposition 4.3.4 *Let $(h, \xi) \in \mathbb{R}_+ \times \mathbb{R}$. Then*

(1) $\mathcal{E}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator;

(2) $\mathcal{E}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a Lipschitz continuous operator with Lipschitz constant $l = 2$ and $\mathcal{E}_{h,\xi}(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$;

(3) $\mathcal{E}_{h,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ extends to a Lipschitz continuous hysteresis operator $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ with Lipschitz constant $l = 2$;

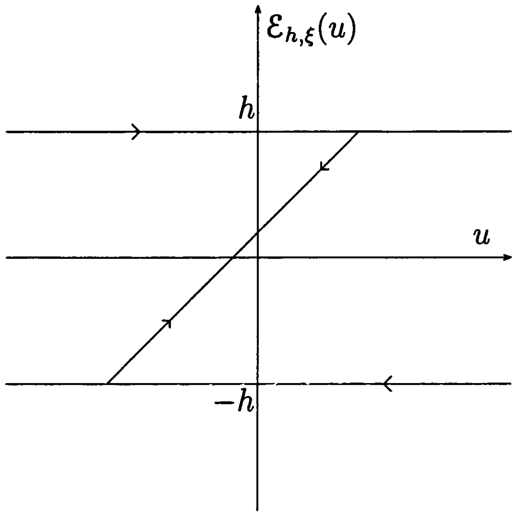

Figure 10: Elastic-plastic hysteresis

(4) $\mathcal{B}_{h,\xi}(u) + \mathcal{E}_{h,\xi}(u) = u$ *for all* $u \in C(\mathbb{R}_+, \mathbb{R})$.

Remark 4.3.5 $l = 2$ is the smallest possible Lipschitz constant for $\mathcal{E}_{h,\xi}$. To illustrate this, consider $u, v \in C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ defined by

$$u(t) = \begin{cases} t + \xi & \text{for } t \in [0, h], \\ h + \xi & \text{for } t > h, \end{cases}$$

$$v(t) = \begin{cases} t + \xi & \text{for } t \in [0, 3h/2], \\ 3h - t + \xi & \text{for } t \in (3h/2, 5h/2], \\ h/2 + \xi & \text{for } t > 5h/2. \end{cases}$$

Then $\sup_{t \in \mathbb{R}_+} |u(t) - v(t)| = h/2$ and $\sup_{t \in \mathbb{R}_+} |(\mathcal{E}_{h,\xi}(u))(t) - (\mathcal{E}_{h,\xi}(v))(t)| = h$. $\diamond$

Proof of Proposition 4.3.4: Statement (1) and Lipschitz continuity (with Lipschitz constant $l = 2$) in statement (2), follow from (4.24) and Proposition 4.3.2, parts (1) and (2). Then by Lemma 4.2.1, $\mathcal{E}_{h,\xi}(C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Statement (3) follows from statements (1) and (2) combined with Proposition 4.2.2. Finally statement (4) follows from (4.24). $\square$

Preisach Operators

All the hysteresis operators considered so far model relatively simple hysteresis loops. The Preisach operator, introduced below, represents a far more general type of hysteresis which for certain input functions exhibits nested loops in the corresponding input-output graphs.

A measure $\mu \in \mathcal{M}(\mathbb{R}_+)$ is called *locally finite* if $|\mu|(S) < \infty$ for all compact sets $S \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of μ.[†] In the following, let $\mathcal{M}_{\text{lf}}(\mathbb{R}_+)$ denote the set of all locally finite signed Borel measures on $\mathbb{R}_+$. Recall that the Lebesgue measure on $\mathbb{R}_+$ is denoted by μ_L.

We define the set of Preisach memory curves

$$\begin{aligned}\Pi := \{\zeta \in C(\mathbb{R}_+, \mathbb{R}) \mid |\zeta(h_1) - \zeta(h_2)| \le |h_1 - h_2| \; \forall \, h_1, h_2 \in \mathbb{R}_+, \\ \zeta \text{ has compact support}\}.\end{aligned}$$

For given $\zeta \in \Pi$, the *Preisach operator* $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, is defined by

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{h,\zeta(h)}(u))(t)} w(h,s)\, ds\, d\mu(h) + w_0 \,, \tag{4.25}$$

where $\mu \in \mathcal{M}_{\text{lf}}(\mathbb{R}_+)$, $w \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$ and $w_0 \in \mathbb{R}$. It is clear that for fixed $\zeta \in \Pi$, $u \in C(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$, the map

$$\psi : \mathbb{R}_+ \to \mathbb{R}, \quad h \mapsto (\mathcal{B}_{h,\zeta(h)}(u))(t)\,,$$

is in Π: using Lemma 4.3.3, ψ is globally Lipschitz with Lipschitz constant 1, and as a direct consequence of the definition of the backlash operator, ψ also has compact support. Consequently, the right-hand side of (4.25) is finite for all $u \in C(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$.

The causality and rate independence of $\mathcal{P}_\zeta$ follow immediately from the causality and rate independence of $\mathcal{B}_{h,\xi}$ and hence $\mathcal{P}_\zeta$ is a hysteresis operator in the sense of Definition 4.1.1.

The proof of the following lemma follows immediately from [4], pp. 58–60.

Lemma 4.3.6 *Let $\mu \in \mathcal{M}_{\text{lf}}(\mathbb{R}_+)$, $w \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$ and $w_0 \in \mathbb{R}$. Suppose that $\lambda := \int_0^\infty \sup_{s\in\mathbb{R}} |w(h,s)|\, d|\mu|(h) < \infty$. Then for all $\zeta \in \Pi$, the Preisach operator $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, defined by (4.25), is Lipschitz continuous with Lipschitz constant λ and for $u \in AC(\mathbb{R}_+, \mathbb{R})$*

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty w(h, (\mathcal{B}_{h,\zeta(h)}(u))(t))(\mathcal{B}_{h,\zeta(h)}(u))'(t)\, d\mu(h)\,, \quad \text{a.e. } t \in \mathbb{R}_+\,,$$

where $'$ denotes differentiation with respect to t.

Remark 4.3.7 Let $\zeta \in \Pi$ and $u \in AC(\mathbb{R}_+, \mathbb{R})$. It is implicit in Lemma 4.3.6 that

[†] If $\mu \in \mathcal{M}(\mathbb{R}_+)$ is locally finite, then it follows that the measure $|\mu|$ is regular, and hence that μ is a signed Radon measure, see [11], pp. 211–222.

for μ_L-almost every $t \in \mathbb{R}_+$, $(\mathcal{B}_{h,\zeta(h)}(u))'(t)$ exists for $|\mu|$-almost every $h \in \mathbb{R}_+$. This result is proved in [4], Lemma 2.4.8. $\diamond$

As an example, we consider the operator $\mathcal{P}_\zeta$ obtained by setting $\zeta \equiv 0$, $\mu = \mu_L$, $w_0 = 0$ and $w \equiv 2 \cdot \chi_{[0,5]\times[0,5]}$. This operator is illustrated in Figure 11.

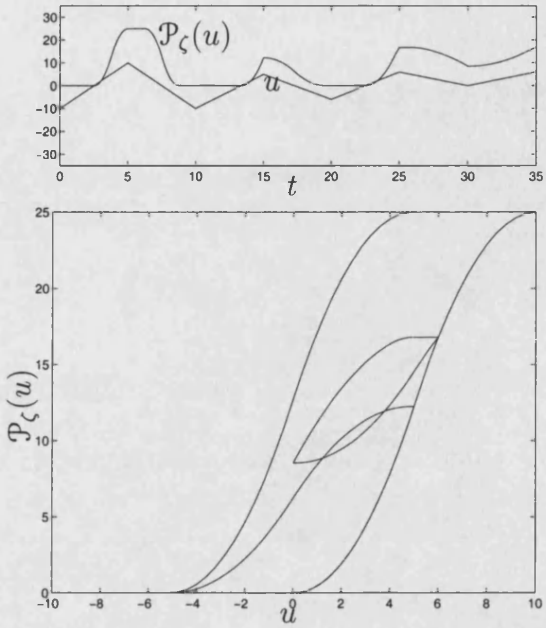

Figure 11: Example of Preisach hysteresis

If we set $w \equiv 1$ and $w_0 = 0$ in the definition of the Preisach operator (i.e. (4.25)), we obtain, the *Prandtl operator*

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty (\mathcal{B}_{h,\zeta(h)}(u))(t)\, d\mu(h)\,, \quad \forall\, u \in C(\mathbb{R}_+, \mathbb{R})\,, \ \forall\, t \in \mathbb{R}_+\,, \qquad (4.26)$$

where $\zeta \in \Pi$ and $\mu \in \mathcal{M}_{\mathrm{lf}}(\mathbb{R}_+)$, cf. [4], pp. 54. For example, defining the measure μ by $\mu(E) = \int_E (\sin(\pi h) + 1)\chi_{[0,10]}(h) dh$ and setting $\zeta \equiv 0$ yields the operator illustrated below in Figure 12.

Finally, we introduce an important subclass of Prandtl operators. Let $p \in L^1(\mathbb{R}_+, \mathbb{R})$ and $\mu = \left(\int_0^\infty p(h)\, dh\right) \delta_0 - p\mu_L$ in (4.26). Then we obtain, for $\zeta \in \Pi$,

$$(\mathcal{P}_\zeta(u))(t) = \int_0^\infty p(h)(\mathcal{E}_{h,\zeta(h)}(u))(t)\, dh\,, \quad \forall\, u \in C(\mathbb{R}_+, \mathbb{R})\,, \ \forall\, t \in \mathbb{R}_+\,, \qquad (4.27)$$

where we have used Proposition 4.3.4, part (4) and the fact that for all $\xi \in \mathbb{R}$ and $u \in C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{B}_{0,\xi}(u) = u$.

For example, setting $p = \chi_{[0,5]}$ and $\zeta \equiv 0$ will produce the operator represented below in Figure 13.

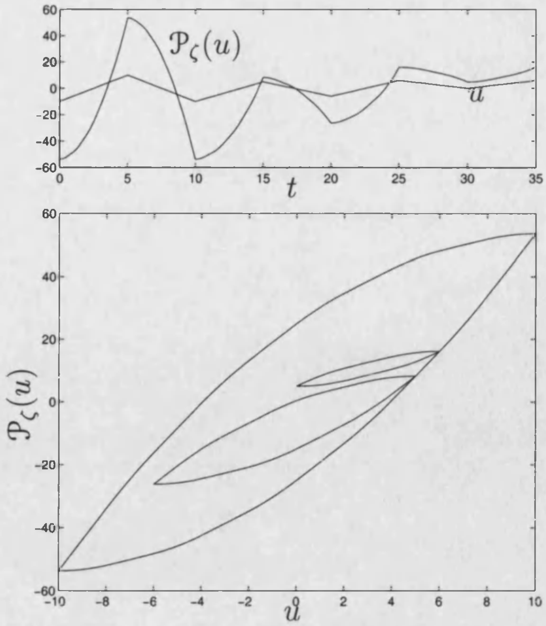

Figure 12: Example of Prandtl hysteresis (4.26)

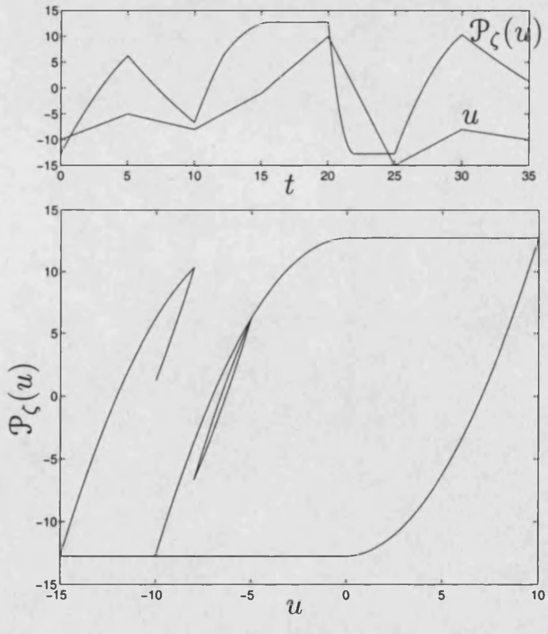

Figure 13: Example of Prandtl hysteresis (4.27)

4.4 Extending hysteresis operators defined on $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ to spaces of piecewise continuous functions

In this section we extend hysteresis operators defined on $C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ to spaces of piecewise continuous functions. This allows us to consider step inputs to hysteresis operators which will prove useful in the context of sampled-data control. Let $NPC(\mathbb{R}_+,\mathbb{R}) \subset PC(\mathbb{R}_+,\mathbb{R})$ denote the space of all *normalised piecewise continuous* functions $u : \mathbb{R}_+ \to \mathbb{R}$, that is u is piecewise continuous and is right-continuous or left-continuous at each point $t \in \mathbb{R}_+$. In particular, if $u \in NPC(\mathbb{R}_+,\mathbb{R})$, then u is right-continuous at $t = 0$. The set of all piecewise monotone functions $u \in NPC(\mathbb{R}_+,\mathbb{R})$ is denoted by $NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, whilst $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ denotes the set of all ultimately constant $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$. We note that $NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ and $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ both satisfy (F1) and (F2) (see beginning of Section 4.1). For $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, we define $u(\infty) := \lim_{\tau\to\infty} u(\tau)$.

Lemma 4.4.1 *Let $u \in NPC(\mathbb{R}_+,\mathbb{R})$, $f \in \mathcal{T}$ and $t > 0$. Define*

$$\tau = \begin{cases} \max f^{-1}(\{t\}) & \text{if } u(t-) = u(t)\,, \\ \min f^{-1}(\{t\}) & \text{if } u(t-) \neq u(t)\,. \end{cases}$$

Then $(u \circ f)(\tau+) = u(t+)$ and $(u \circ f)(\tau-) = u(t-)$.

Proof: Since f is continuous, non-decreasing and surjective, for all $t \in \mathbb{R}_+$, $f^{-1}(\{t\})$ is a compact interval and therefore τ is well defined. We consider two cases.

CASE 1. Suppose that $u(t-) = u(t)$. Then, $f(\tau + h) > t$ for all $h > 0$ and so $(u \circ f)(\tau+) = u(f(\tau)+) = u(t+)$. Moreover, if $f^{-1}(\{t\})$ is a singleton, we have $f(\tau - h) < t$ for all $h \in (0, \tau]$ and so $(u \circ f)(\tau-) = u(f(\tau)-) = u(t-)$. If $f^{-1}(\{t\})$ is not a singleton, we have $f(\tau - h) = t$ for all sufficiently small $h > 0$ and so $(u \circ f)(\tau-) = u(t) = u(t-)$.

CASE 2. Suppose that $u(t-) \neq u(t)$. Then, since $u \in NPC(\mathbb{R}_+,\mathbb{R})$, it follows that $u(t+) = u(t)$. Adopting an argument similar to that in Case 1 yields the claim. □

We define the map $\rho : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R})$ by

$$\rho(u) = (u(t_0), u(t_1-), u(t_1+), u(t_2-), \ldots, u(t_m-), u(t_m+), u(\infty), u(\infty), \ldots)\,,$$

where $0 = t_0 < t_1 < \ldots < t_m$ is the standard monotonicity partition of u. Let $\tau > 0$. We define the *prolongation operator* $P_\tau : F(\mathbb{Z}_+,\mathbb{R}) \to C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ by letting $P_\tau u$ be the linear interpolant for the values $(P_\tau u)(i\tau) = u(i)$. Moreover, we introduce the operator

$$\tilde{R} : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R}) \to F^{\mathrm{uc}}(\mathbb{Z}_+,\mathbb{R})\,, \quad u \mapsto R((P_\tau \circ \rho)(u))\,.$$

The function u, shown in Figure 14, is a normalized piecewise continuous function

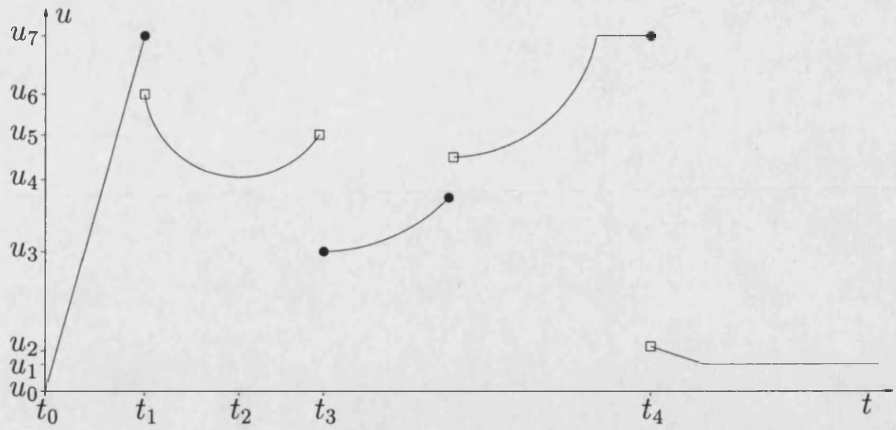

Figure 14: Example of a function in $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

which is piecewise monotone and ultimately constant, so $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. It has standard monotonicity partition $0 = t_0 < t_1 < t_2 < t_3 < t_4$,

$$\rho(u) = (u_0, u_7, u_6, u_4, u_4, u_5, u_3, u_7, u_2, u_1, u_1, u_1, \ldots),$$

and

$$\tilde{R}(u) = (u_0, u_7, u_4, u_5, u_3, u_7, u_1, u_1, u_1, \ldots).$$

Clearly, for any $\tau > 0$

$$R \circ P_\tau \circ R = R, \tag{4.28}$$

and using (4.28) it is easy to show that $R \circ P_\tau \circ \tilde{R} = \tilde{R}$.

Lemma 4.4.2 *$\tilde{R}$ is an extension of R, the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$ and*

$$\tilde{R}(u \circ f) = \tilde{R}(u), \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}), \quad \forall\, f \in \mathfrak{T}. \tag{4.29}$$

Proof: Let $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, then, since $u(t+) = u(t-)$ for all $t > 0$,

$$\tilde{R}(u) = R((P_\tau \circ \rho)(u)) = R(u),$$

showing that $\tilde{R}$ is an extension of R. Let $\tau_1, \tau_2 > 0$ and $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Clearly, $R((P_{\tau_1} \circ \rho)(u)) = R((P_{\tau_2} \circ \rho)(u))$, from which it follows that the definition of $\tilde{R}$ does not depend upon the choice of $\tau > 0$.

Finally, let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $f \in \mathfrak{T}$ and let $0 = t_0 < t_1 < \ldots < t_m$ be the standard monotonicity partition of u. Define $\tau_0 := 0$ and for $i = 1, \ldots, m$

$$\tau_i := \begin{cases} \max f^{-1}(\{t_i\}) & \text{if } u(t_i-) = u(t_i), \\ \min f^{-1}(\{t_i\}) & \text{if } u(t_i-) \neq u(t_i). \end{cases}$$

Then $0 = \tau_0 < \tau_1 < \ldots < \tau_m$ is the standard monotonicity partition of $u \circ f$ and by Lemma 4.4.1, $(u \circ f)(\tau_i \pm) = u(t_i \pm)$ for $i = 0, 1, \ldots, m$. Hence, $\rho(u) = \rho(u \circ f)$ and therefore, $\tilde{R}(u) = \tilde{R}(u \circ f)$, showing that (4.29) holds. □

For any rate independent $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ we define

$$\tilde{\varphi} : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}, \quad u \mapsto \varphi((P_\tau \circ \tilde{R})(u)), \tag{4.30}$$

where $\tau > 0$. We show that the definition of $\tilde{\varphi}$ does not depend on τ. To this end, let $\tau_1, \tau_2 > 0$ and $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then, clearly there exists $f \in \mathcal{T}$ such that $(P_{\tau_1} \circ \tilde{R})(u) = (P_{\tau_2} \circ \tilde{R})(u) \circ f$ and therefore by the rate independence of φ

$$\varphi((P_{\tau_1} \circ \tilde{R})(u)) = \varphi((P_{\tau_2} \circ \tilde{R})(u)) .$$

Lemma 4.4.3 *Let $\varphi : C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}$ be rate independent and define $\tilde{\varphi}$ by (4.30). Then*

(1) $\tilde{\varphi}$ is an extension of φ, i.e.

$$\tilde{\varphi}(u) = \varphi(u), \quad \forall\, u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R});$$

(2) for any $\tau > 0$

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \rho)(u)), \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R});$$

(3) for $u, v \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$

$$\tilde{R}(u) = \tilde{R}(v) \quad \Longrightarrow \quad \tilde{\varphi}(u) = \tilde{\varphi}(v);$$

(4) $\tilde{\varphi}$ is rate independent, i.e.

$$\tilde{\varphi}(u \circ f) = \tilde{\varphi}(u), \quad \forall\, u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}), \quad \forall\, f \in \mathcal{T}.$$

Proof: Let $\tau > 0$ and $u \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Clearly, $R(u) = R((P_\tau \circ R)(u))$, and so using Lemma 4.1.4, there exist $f, g \in \mathcal{T}$ such that $u \circ f = (P_\tau \circ R)(u) \circ g$. Thus the rate independence of φ in combination with Lemma 4.4.2 gives

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \tilde{R})(u)) = \varphi((P_\tau \circ R)(u) \circ g) = \varphi(u \circ f) = \varphi(u),$$

which is statement (1). To prove statement (2), let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. By definition $\tilde{R}(u) = R((P_\tau \circ \rho)(u))$ and therefore,

$$(P_\tau \circ \tilde{R})(u) = P_\tau(R((P_\tau \circ \rho)(u))) .$$

Thus, invoking (4.28)

$$R((P_\tau \circ \tilde{R})(u)) = R(P_\tau(R((P_\tau \circ \rho)(u)))) = R((P_\tau \circ \rho)(u)),$$

and so using the rate independence of φ and Lemma 4.1.4

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \tilde{R})(u)) = \varphi((P_\tau \circ \rho)(u)).$$

For statement (3), let $u, v \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Suppose that $\tilde{R}(u) = \tilde{R}(v)$; then by the definition of $\tilde{R}$, $R((P_\tau \circ \rho)(u)) = R((P_\tau \circ \rho)(v))$. Since $(P_\tau \circ \rho)(u), (P_\tau \circ \rho)(v) \in C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ it follows from an application of Lemma 4.1.4, the rate independence of φ and statement (2) that

$$\tilde{\varphi}(u) = \varphi((P_\tau \circ \rho)(u)) = \varphi((P_\tau \circ \rho)(v)) = \tilde{\varphi}(v).$$

Statement (4) follows immediately from (4.29) and statement (3). □

Definition 4.4.4 For a hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ define

$$\tilde{\Phi} : NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R}),$$

by setting

$$(\tilde{\Phi}(u))(t) = \tilde{\varphi}(\mathbf{Q}_t u), \quad \forall t \in \mathbb{R}_+, \tag{4.31}$$

where φ is the representing functional of Φ and $\tilde{\varphi}$ is the extension of φ to $NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (4.30). ◇

Theorem 4.4.5 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and let $\tilde{\Phi} : NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be defined by (4.31). Then*

(1) $\tilde{\Phi}$ is an extension of Φ;

(2) $\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$;

(3) for $u, v \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$

$$\tilde{R}(\mathbf{Q}_t u) = \tilde{R}(\mathbf{Q}_t v) \quad \Longrightarrow \quad (\tilde{\Phi}(u))(t) = (\tilde{\Phi}(v))(t);$$

(4) $\tilde{\Phi}(\mathcal{S}^r) \subset \mathcal{S}^r$ and $\tilde{\Phi}(\mathcal{S}^r_\tau) \subset \mathcal{S}^r_\tau$.

Proof: Statement (1) is clear since $\tilde{\varphi}$ is an extension of φ and thus $\tilde{\Phi}$ is an extension of Φ. By Lemma 4.4.3, part (4), $\tilde{\varphi}$ is rate independent. Therefore, by Theorem 4.1.2, $\tilde{\Phi}$ is a hysteresis operator with representing functional $\tilde{\varphi}$, showing that statement (2) holds. Statement (3) follows from the definition of $\tilde{\Phi}$ and Lemma 4.4.3, part (3). Since $\mathcal{S}^r_\tau \subset \mathcal{S}^r \subset NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, statement (4) follows from statement (2) combined with Corollary 4.1.3. □

In the following we define continuous, piecewise monotone "approximations" u_1, u_2, u_3, ... of a given function $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $\Phi(u_k)$ "approximates" $\tilde{\Phi}(u)$ as $k \to \infty$. Let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of u and set $\tau_0 := 0$. For each $k \in \mathbb{Z}_+$, define

$$\varepsilon_k := \frac{1}{k+2} \min_{1 \le i \le n} (\tau_i - \tau_{i-1}). \tag{4.32}$$

For each $k \in \mathbb{Z}_+$, define an operator $C_k : NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by setting:

(1) if $t \in [\tau_j - \varepsilon_k, \tau_j)$ and u is right-continuous at τ_j, then

$$(C_k(u))(t) = \begin{cases} \text{linear interpolant of } u(\tau_j - \varepsilon_k) \text{ and } u(\tau_j-), & t \in [\tau_j - \varepsilon_k, \tau_j - \varepsilon_k/2], \\ \text{linear interpolant of } u(\tau_j-) \text{ and } u(\tau_j), & t \in [\tau_j - \varepsilon_k/2, \tau_j); \end{cases}$$

(2) if $t \in (\tau_j, \tau_j + \varepsilon_k]$ and u is left-continuous at τ_j, then

$$(C_k(u))(t) = \begin{cases} \text{linear interpolant of } u(\tau_j) \text{ and } u(\tau_j+), & t \in (\tau_j, \tau_j + \varepsilon_k/2], \\ \text{linear interpolant of } u(\tau_j+) \text{ and } u(\tau_j + \varepsilon_k), & t \in [\tau_j + \varepsilon_k/2, \tau_j + \varepsilon_k]; \end{cases}$$

(3) $(C_k(u))(t) = u(t)$ otherwise

(see Figure 15 for an illustration).

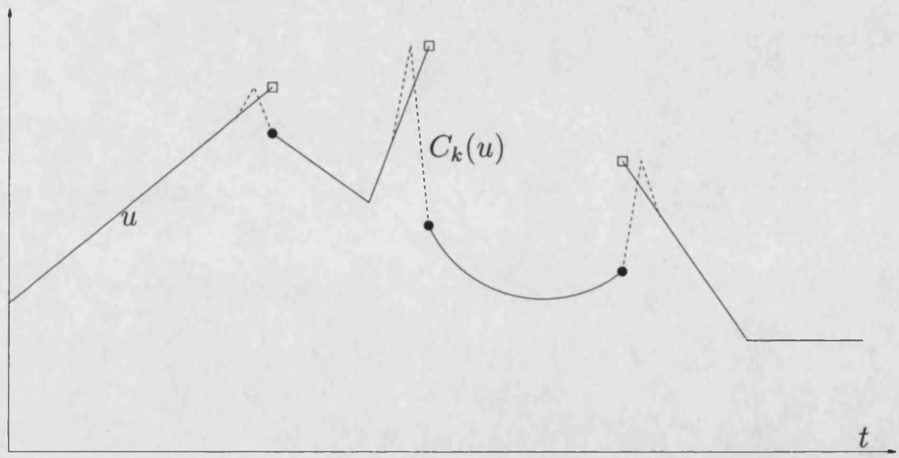

Figure 15: Example of a function $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and its approximation $C_k(u)$

Lemma 4.4.6 *Let $u \in NPC^{\mathrm{uc}}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Then*

(1) *for any* $t \in \mathbb{R}_+$, *there exists* $l > 0$ *such that*

$$\tilde{R}(\mathbf{Q}_t\, u) = R(\mathbf{Q}_t\, C_k(u))\,, \quad \forall\, k \geq l\,;$$

(2) *for any* $t_2 > t_1 \geq 0$, *if* u *is continuous on* $[t_1, t_2]$, *then there exists* $l > 0$ *such that*

$$\tilde{R}(\mathbf{Q}_s\, u) = R(\mathbf{Q}_s\, C_k(u))\,, \quad \forall\, s \in [t_1, t_2]\,, \quad \forall\, k \geq l\,.$$

Proof: Let $u \in NPC_{\mathrm{pm}}^{\mathrm{uc}}(\mathbb{R}_+, \mathbb{R})$, let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of u and let $0 = t_0 < t_1 < \ldots < t_m$ be the standard monotonicity partition of u. Define ε_k by (4.32).

For statement (1), let $t \in \mathbb{R}_+$ and choose $l > 0$ such that

$$\varepsilon_l < \min\{|t_i - \tau_j| \,|\, 1 \leq i \leq m, 1 \leq j \leq n, t_i \neq \tau_j\}\,,$$

and

$$\varepsilon_l < \min\{|t - \tau_j| \,|\, 1 \leq j \leq n, t \neq \tau_j\}\,.$$

Then $\tilde{R}(\mathbf{Q}_t\, u) = R(\mathbf{Q}_t\, C_k(u))$ for all $k \geq l$.

To prove statement (2), let $t_2 > t_1 \geq 0$ be such that u is continuous on $[t_1, t_2]$. Hence, there exists $l_1 > 0$ such that

$$C_k(u)|_{[t_1,t_2]} = u|_{[t_1,t_2]}\,, \quad \forall\, k \geq l_1\,. \tag{4.33}$$

Moreover, by statement (1), there exists $l_2 > 0$ such that

$$\tilde{R}(\mathbf{Q}_{t_1}\, u) = R(\mathbf{Q}_{t_1}\, C_k(u))\,, \quad \forall\, k \geq l_2\,. \tag{4.34}$$

Hence, by (4.33) and (4.34) statement (2) holds for $l := \max(l_1, l_2)$. $\square$

For $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $t > 0$ and $\varepsilon > 0$, we define

$$J_\varepsilon(u, t) := \cup_{i=1}^{n}(\tau_i - \varepsilon, \tau_i + \varepsilon) \quad \text{and} \quad d(u, t) := \min_{1 \leq i \leq n-1}(\tau_{i+1} - \tau_i)/2\,,$$

where $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of $\mathbf{Q}_t\, u$.

Proposition 4.4.7 *Let* $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ *be a hysteresis operator and* $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. *Then*

(1) *for all* $t \in \mathbb{R}_+$, *there exists* $l > 0$ *such that*

$$(\tilde{\Phi}(u))(t) = (\Phi(C_k(\mathbf{Q}_t\, u)))(t)\,, \quad \forall\, k \geq l\,;$$

(2) if for $t_3 \geq t_2 > t_1 \geq 0$, u is continuous on $[t_1, t_2]$, then there exists $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(\mathbf{Q}_{t_3} u)))(s)\,, \quad \forall\, s \in [t_1, t_2]\,, \quad \forall\, k \geq l\,;$$

(3) for all $t \in \mathbb{R}_+$ and all $\varepsilon \in (0, d(u,t))$, there exists $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(\mathbf{Q}_t u)))(s)\,, \quad \forall\, s \in [0,t) \setminus J_\varepsilon(u,t)\,, \quad \forall\, k \geq l\,.$$

Proof: Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Statement (1) follows from Theorem 4.4.5 and Lemma 4.4.6, part (1), and statement (2) follows from Theorem 4.4.5 and Lemma 4.4.6, part (2). For statement (3), let $t \in \mathbb{R}_+$, $\varepsilon \in (0, d(u,t))$ and let $0 < \tau_1 < \tau_2 < \ldots < \tau_n$ denote the points of discontinuity of $\mathbf{Q}_t u$. Clearly u is continuous on $[\tau_i + \varepsilon, \tau_{i+1} - \varepsilon]$ for $1 \leq i \leq n-1$. Therefore by statement (2) and the causality of Φ, there exists $l_i > 0$ such that for $1 \leq i \leq n-1$

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(\mathbf{Q}_t u)))(s)\,, \quad \forall\, s \in [\tau_i + \varepsilon, \tau_{i+1} - \varepsilon]\,, \quad \forall\, k \geq l_i\,.$$

To conclude the proof, we distinguish between two cases: $\tau_n + \varepsilon < t$ and $\tau_n + \varepsilon \geq t$.

If $\tau_n + \varepsilon < t$, then u is continuous on $[\tau_n + \varepsilon, t]$ and therefore again by statement (2), there exists $l_n > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(\mathbf{Q}_t u)))(s)\,, \quad \forall\, s \in [\tau_n + \varepsilon, t]\,, \quad \forall\, k \geq l_n\,.$$

If $\tau_n + \varepsilon \geq t$, then set $l_n := 0$.

In both cases define $l := \max_{1 \leq i \leq n} l_i$ and statement (3) then follows. □

The following example shows that $\tilde{\Phi}$, defined by (4.31), is not the unique hysteresis operator extending a given hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$.

Example 4.4.8 Define $Z_e : NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(Z_e(u))(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sum\limits_{0 < \tau \leq t} (u(\tau) - u(\tau-)) & \text{if } t > 0\,. \end{cases}$$

Clearly, Z_e is a causal extension of the trivial operator

$$Z : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})\,, \quad u \mapsto 0\,.$$

We show that Z_e is rate independent. Let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, $f \in \mathcal{T}$, $t > 0$ and let $0 < t_1 < \ldots < t_m \leq f(t)$ be the points at which $\mathbf{Q}_{f(t)}\, u$ is not left-continuous. Define, for $i = 1, \ldots, m$, $\tau_i := \min f^{-1}(\{t_i\})$. Then $0 < \tau_1 < \ldots < \tau_m \leq t$ are the points at which $\mathbf{Q}_t\,(u \circ f)$ is not left-continuous. By Lemma 4.4.1, $(u \circ f)(\tau_i-) = u(t_i-)$ for $i = 1, \ldots, m$, and thus

$$\begin{aligned}
(Z_e(u))(f(t)) &= \sum_{0<\tau\leq f(t)} (u(\tau) - u(\tau-)) \\
&= \sum_{i=1}^{m} (u(t_i) - u(t_i-)) \\
&= \sum_{i=1}^{m} ((u \circ f)(\tau_i) - (u \circ f)(\tau_i-)) \\
&= \sum_{0<\tau\leq t} ((u \circ f)(\tau) - (u \circ f)(\tau-)) \\
&= (Z_e(u \circ f))(t)\,,
\end{aligned}$$

showing that Z_e is rate independent. Therefore Z_e is a hysteresis operator, but $Z_e \neq \tilde{Z} = 0$. It follows that if $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator, then $\tilde{\Phi} + Z_e$, as well as $\tilde{\Phi}$, are hysteresis operators which extend Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. $\diamond$

The following corollary says that, given a hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$, then $\tilde{\Phi}$ is the unique operator extending Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and satisfying statement (3) of Theorem 4.4.5.

Corollary 4.4.9 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Suppose that Φ_e is an extension of Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. If for all $u, v \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$*

$$\tilde{R}(\mathbf{Q}_t\, u) = \tilde{R}(\mathbf{Q}_t\, v) \quad \Longrightarrow \quad (\Phi_e(u))(t) = (\Phi_e(v))(t)\,, \tag{4.35}$$

then $\Phi_e = \tilde{\Phi}$.

Proof: Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator, Φ_e be an extension of Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ satisfying (4.35), let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$. By Lemma 4.4.6, part (1), for all sufficiently large k, we have

$$\tilde{R}(\mathbf{Q}_t\, u) = \tilde{R}(\mathbf{Q}_t\, C_k(\mathbf{Q}_t\, u))\,.$$

Hence, by (4.35), for all sufficiently large k

$$(\Phi_e(u))(t) = (\Phi_e(C_k(\mathbf{Q}_t\, u)))(t) = (\Phi(C_k(\mathbf{Q}_t\, u)))(t)\,.$$

It follows from Proposition 4.4.7, part (1), that $(\Phi_e(u))(t) = (\tilde{\Phi}(u))(t)$. □

Corollary 4.4.10 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator. Assume that $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Then*

$$\tilde{\Phi}(NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset NPC(\mathbb{R}_+, \mathbb{R}),$$

and for $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, right-continuity of u at $t \geq 0$ (respectively, left-continuity at $t > 0$) implies right-continuity of $\tilde{\Phi}(u)$ at $t \geq 0$ (respectively, left-continuity at $t > 0$).

Proof: Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and assume that $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. Let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We proceed in four steps.

STEP 1. Let us suppose that u is right-continuous at $t \geq 0$, then there exists $\tau > t$ such that u is continuous on $[t, \tau]$. So by Proposition 4.4.7, part (2), there exist $l > 0$ such that

$$(\tilde{\Phi}(u))(s) = (\Phi(C_k(\mathbf{Q}_\tau u))))(s), \quad \forall s \in [t, \tau], \quad \forall k \geq l.$$

Since by assumption $\Phi(C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$, $\Phi(C_k(\mathbf{Q}_\tau u))$ is a continuous function and so $\tilde{\Phi}(u)$ is right-continuous at t.

STEP 2. Similarly, if u is left-continuous at $t > 0$, then it is easy to show that $\tilde{\Phi}(u)$ is left-continuous at t.

STEP 3. Assume that u is left-continuous at $t > 0$. We now show that the right limit $\lim_{s \downarrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. To this end, define $w = u$ on $\mathbb{R}_+ \setminus \{t\}$ and $w(t) = \lim_{s \downarrow t} u(s)$. Thus w is right-continuous at t. Now $\tilde{R}(\mathbf{Q}_\tau u) = \tilde{R}(\mathbf{Q}_\tau w)$ for all $\tau \in \mathbb{R}_+ \setminus \{t\}$ and therefore by Theorem 4.4.5, part (3), $\tilde{\Phi}(u) = \tilde{\Phi}(w)$ on $\mathbb{R}_+ \setminus \{t\}$. Thus

$$\lim_{s \downarrow t}(\tilde{\Phi}(u))(s) = \lim_{s \downarrow t}(\tilde{\Phi}(w))(s) = (\tilde{\Phi}(w))(t),$$

since $\tilde{\Phi}(w)$ is right-continuous at t by Step 1.

STEP 4. Similarly, if u is right-continuous at $t > 0$, then it is easy to show that the left limit $\lim_{s \uparrow t}(\tilde{\Phi}(u))(s)$ exists and is finite. □

We end this section by considering the extension of the standard backlash (or play) operator introduced in the previous section.

Example 4.4.11 By Theorem 4.4.5, the extension $\tilde{\mathfrak{B}}_{h,\xi}$, given by (4.31), of $\mathfrak{B}_{h,\xi}$ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ is a hysteresis operator. By Corollary 4.4.10, we know that

$\tilde{\mathfrak{B}}_{h,\xi}(NPC_{\rm pm}(\mathbb{R}_+,\mathbb{R})) \subset NPC(\mathbb{R}_+,\mathbb{R})$. It is shown in the Appendices (see Appendix 4) that $\tilde{\mathfrak{B}}_{h,\xi}$ can be written recursively as

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = \begin{cases} b_h(u(0),\xi) & \text{for } t = 0\,, \\ b_h(u(t),(\tilde{\mathfrak{B}}_{h,\xi}(u))(0)) & \text{for } 0 < t < t_1\,, \\ b_h(u(t_i),(\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-)) & \text{for } t = t_i,\ i \in \mathbb{N}\,, \\ b_h(u(t),b_h(u(t_i+),(\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-))) & \text{for } t_i < t < t_{i+1},\ i \in \mathbb{N}\,, \end{cases} \tag{4.36}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ is such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval (t_i, t_{i+1}). ◇

4.5 Discrete-time hysteresis operators and discretizations of continuous-time hysteresis operators

We call a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ a *(discrete-time) time transformation* if f is surjective and non-decreasing. We denote the set of all discrete-time time transformations $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $\mathfrak{T}^d$. For each $k \in \mathbb{Z}_+$, we define a (discrete-time) projection operator $\mathbf{Q}_k^d : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ by

$$(\mathbf{Q}_k^d u)(n) = \begin{cases} u(n) & \text{for } n \in [0,k] \cap \mathbb{Z}_+\,, \\ u(k) & \text{for } m \in \mathbb{Z}_+ \setminus [0,k]\,. \end{cases}$$

We call an operator $\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ *causal* if for all $u, v \in F(\mathbb{Z}_+,\mathbb{R})$ and all $k \in \mathbb{Z}_+$ with $u(n) = v(n)$ for all $n \in [0,k] \cap \mathbb{Z}_+$ it follows that $(\Phi(u))(n) = (\Phi(v))(n)$ for all $n \in [0,k] \cap \mathbb{Z}_+$. An operator $\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ is called *rate independent* if

$$(\Phi(u \circ f))(n) = (\Phi(u))(f(n))\,, \quad \forall\, u \in F(\mathbb{Z}_+,\mathbb{R})\,, \quad \forall\, f \in \mathfrak{T}^d, \quad \forall\, n \in \mathbb{Z}_+\,.$$

Definition 4.5.1 An operator $\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ is called a *(discrete-time) hysteresis operator* if Φ is causal and rate independent. ◇

Recall that $F^{\rm uc}(\mathbb{Z}_+,\mathbb{R})$ denotes the set of all ultimately constant $u \in F(\mathbb{Z}_+,\mathbb{R})$. A functional $\varphi : F^{\rm uc}(\mathbb{Z}_+,\mathbb{R}) \to \mathbb{R}$ is called *rate independent* if

$$\varphi(u \circ f) = \varphi(u)\,, \quad \forall\, u \in F^{\rm uc}(\mathbb{Z}_+,\mathbb{R})\,, \quad \forall\, f \in \mathfrak{T}^d.$$

The proof of the following theorem is analogous to the proof of Theorem 4.1.2 and is therefore omitted.

Theorem 4.5.2 *If $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ is a hysteresis operator, then*

(1) for all $u \in F(\mathbb{Z}_+, \mathbb{R})$ and all $k \in \mathbb{Z}_+$

$$(\Phi(\mathbf{Q}_k^d u))(n) = (\Phi(u))(k)\,, \quad \forall\, n \geq k\,;$$

(2) the functional

$$\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}, \quad u \mapsto \lim_{n\to\infty} (\Phi(u))(n)\,, \tag{4.37}$$

is rate independent and satisfies

$$(\Phi(u))(n) = \varphi(\mathbf{Q}_n^d u)\,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,. \tag{4.38}$$

Conversely, if $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ is a rate independent functional, then $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ given by (4.38) is a hysteresis operator and satisfies

$$\lim_{n\to\infty} (\Phi(u))(n) = \varphi(u)\,, \quad \forall\, u \in F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R})\,.$$

For a hysteresis operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$, we call the rate independent functional $\varphi : F^{\mathrm{uc}}(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ defined by (4.37) the *representing functional* of Φ.

Let $\tau > 0$. The *hold operator* $H_\tau : F(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{S}_\tau^r$ is defined by

$$(H_\tau u)(n\tau + t) = u(n)\,, \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,, \quad \forall\, t \in [0, \tau)\,,$$

and the *sampling operator* $S_\tau : F(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$(S_\tau u)(n) = u(n\tau)\,, \quad \forall\, u \in F(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

Definition 4.5.3 For a continuous-time hysteresis operator $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ define $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ by

$$\Phi^d := S_\tau \,\tilde{\Phi}\, H_\tau\,, \tag{4.39}$$

where $\tilde{\Phi}$ is the extension of Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ defined in (4.31). ◇

The definition of Φ^d is independent of the choice of τ due to the rate independence of $\tilde{\Phi}$.

Proposition 4.5.4 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a continuous-time hysteresis operator. Then $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$, defined by (4.39), is a discrete-time hysteresis operator.*

Proof: It is clear that Φ^d is causal. It remains to show that Φ^d is rate independent. Let $u \in F(\mathbb{Z}_+, \mathbb{R})$ and $f \in \mathcal{T}^d$, then $f^c := \tau P_\tau(f) \in \mathcal{T}$ and $(H_\tau u) \circ f^c = H_\tau(u \circ f)$. Hence using the rate independence of $\tilde{\Phi}$,

$$\begin{aligned}(\Phi^d(u \circ f))(n) &= (\tilde{\Phi}(H_\tau(u \circ f)))(n\tau) = (\tilde{\Phi}((H_\tau u) \circ f^c))(n\tau) \\ &= (\tilde{\Phi}(H_\tau u))(f^c(n\tau)) = (\tilde{\Phi}(H_\tau u))(f(n)\tau) = (\Phi^d(u))(f(n)),\end{aligned}$$

showing that Φ^d is rate independent. □

Definition 4.5.5 Let $\mathbb{T} = \mathbb{Z}_+, \mathbb{R}_+$ and $\mathcal{F} \subset F(\mathbb{T}, \mathbb{R})$, $\mathcal{F} \neq \emptyset$; then the *numerical value set* of an operator $\Psi : \mathcal{F} \to F(\mathbb{T}, \mathbb{R})$, NVS Ψ, is defined by

$$\text{NVS}\,\Psi := \{(\Psi(u))(t) \mid u \in \mathcal{F},\, t \in \mathbb{T}\}. \tag{4.40}$$

◇

The following proposition shows that for a continuous-time hysteresis operator Φ defined on $C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$, the numerical value sets of Φ and Φ^d (defined by (4.39)) coincide. This result is important in the context of sampled-data low-gain control of systems subject to input hysteresis (see Chapter 8), but is also of some interest in its own right.

Proposition 4.5.6 *Let $\Phi : C_{\text{pm}}(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ be a continuous-time hysteresis operator and let $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ be defined by (4.39). Then*

$$(\Phi^d(u))(n) = (\Phi(P_\tau u))(n\tau), \quad \forall\, u \in F(\mathbb{Z}_+, \mathbb{R}), \quad \forall\, n \in \mathbb{Z}_+, \tag{4.41}$$

and NVS Φ^d = NVS Φ.

Proof: Let $u \in F(\mathbb{Z}_+, \mathbb{R})$ and $n \in \mathbb{Z}_+$. We note that $\tilde{R}(\mathbf{Q}_{n\tau}\, H_\tau\, u) = R(\mathbf{Q}_{n\tau}\, P_\tau\, u)$ and so by Theorem 4.4.5, parts (1) and (3)

$$(\Phi^d(u))(n) = (\tilde{\Phi}(H_\tau u))(n\tau) = (\Phi(P_\tau u))(n\tau).$$

To prove that NVS Φ^d = NVS Φ, note first, that by (4.41), NVS $\Phi^d \subset$ NVS Φ. To show the reverse inclusion, let $a \in$ NVS Φ. Then there exist $v \in C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$ such that $a = (\Phi(v))(t)$. Set $w := \mathbf{Q}_t\, v \in C_{\text{pm}}^{\text{uc}}(\mathbb{R}_+, \mathbb{R})$. Clearly

$$\mathbf{Q}_{k\tau} w = w, \quad \forall\, k \geq t/\tau.$$

Moreover, $(P_\tau \circ R)(w) \in C_{\text{pm}}^{\text{uc}}(\mathbb{R}_+, \mathbb{R})$ and so there exists $k_0 > 0$ such that

$$\mathbf{Q}_{k\tau}((P_\tau \circ R)(w)) = (P_\tau \circ R)(w), \quad \forall\, k \geq k_0.$$

For $k \geq \max(k_0, t/\tau) =: k_1$ we then have

$$(P_\tau \circ R)(\mathbf{Q}_{k\tau} w) = (P_\tau \circ R)(w) = \mathbf{Q}_{k\tau}((P_\tau \circ R)(w)). \tag{4.42}$$

Let φ be the representing functional of Φ, then for $k \geq k_1$

$$a = (\Phi(w))(t) = (\Phi(w))(k\tau) = \varphi(\mathbf{Q}_{k\tau} w) = \tilde{\varphi}(\mathbf{Q}_{k\tau} w) = \varphi((P_\tau \circ R)(\mathbf{Q}_{k\tau} w)), \tag{4.43}$$

where we have used Theorem 4.1.2, statements (1) and (2) and the fact that $\tilde{\varphi}$ is an extension of φ. Combining Theorem 4.1.2, statement (2), (4.41), (4.42) and (4.43), we obtain for any $k \geq k_1$

$$a = \varphi(\mathbf{Q}_{k\tau}(P_\tau \circ R)(w)) = (\Phi((P_\tau \circ R)(w)))(k\tau) = (\Phi^d(Rw))(k) \in \mathrm{NVS}\, \Phi^d.$$

□

We finally look at the discretization of the (standard) backlash operator.

Example 4.5.7 Let $h \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$. Let $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be the (standard) backlash operator defined by (4.15), where b is given by (4.20). We consider the discretization $\mathcal{B}^d_{h,\xi} : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ given by $\mathcal{B}^d_{h,\xi} = S_\tau \tilde{\mathcal{B}}_{h,\xi} H_\tau$, where $\tilde{\mathcal{B}}_{h,\xi}$ is the extension of $\mathcal{B}_{h,\xi}$ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (4.31). Using (4.41), we see that for all $u \in F(\mathbb{Z}_+, \mathbb{R})$, $\mathcal{B}^d_{h,\xi}(u)$ can be expressed recursively as

$$(\mathcal{B}^d_{h,\xi}(u))(n) = \begin{cases} b_h(u(0), \xi) & \text{for } n = 0, \\ b_h(u(n), (\mathcal{B}^d_{h,\xi}(u))(n-1)) & \text{for } n \in \mathbb{N}, \end{cases} \tag{4.44}$$

where $b_h : \mathbb{R}^2 \to \mathbb{R}$ is given by (4.20). ◇

4.6 Notes and references

In Section 4.1, our treatment of hysteresis operators is strongly influenced by chapter 2 in the book [4] by Brokate and Sprekels. Most of the results in this section can be found in a somewhat different and less general form in chapter 2 of [4] though not always with proof. In contrast to most of the literature (see [4] and [39]), where hysteresis operators act on functions with a finite time horizon, we have considered hysteresis operators acting on functions defined on the infinite interval, $[0, \infty)$. The results of Section 4.2 are new. The models of hysteresis presented in Section 4.3 are all well known: specifically relay hysteresis can be found in [31]; generalized backlash in [16]; elastic-plastic in [4]; and Preisach in [4].

It should be noted though that we have proved some results for the generalized backlash and elastic-plastic operators which are not proved in [16] or [4] (in particular the proofs that the definitions of $\mathcal{B}_\xi(u)$ and $\mathcal{E}_{h,\xi}(u)$ are independent of the choice of partition $(t_i) \in P_u$). Sections 4.4 and 4.5 consist of entirely new material, and form the basis of [20] by Logemann and Mawby. The extension introduced in Section 4.4 and the discretization introduced in Section 4.5 were motivated by our interest in sampled-data control of systems with hysteresis effects. However, we believe that the results of Sections 4.4 and 4.5 are of interest in their own right and therefore many of the results (such as Proposition 4.4.7 and Corollary 4.4.10) are more general than is needed in Chapter 8 where we consider sampled-data control of systems with hysteresis effects.

Chapter 5

Classes of hysteresis operators

5.1 Classes of continuous-time hysteresis operators

Let $u \in C(\mathbb{R}_+, \mathbb{R})$. The function u is called *ultimately non-decreasing* if there exists $T \in \mathbb{R}_+$ such that u is non-decreasing on $[T, \infty)$; u is said to be *approximately ultimately non-decreasing*, if for all $\varepsilon > 0$, there exists an ultimately non-decreasing function $v \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$\sup_{t \in \mathbb{R}_+} |u(t) - v(t)| \leq \varepsilon .$$

For the rest of this chapter let $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$. For $\alpha \in \mathbb{R}_+$, define $\mathcal{C}_\alpha := \{f|_{[0,\alpha]} \,|\, f \in \mathcal{C}\}$. For $\alpha \in \mathbb{R}_+$, $w \in \mathcal{C}_\alpha$ and $\delta_1, \delta_2 > 0$, we define $\mathcal{C}(w; \delta_1, \delta_2)$ to be the set of all $u \in \mathcal{C}$ such that

$$u(t) = w(t)\,, \ \forall\, t \in [0, \alpha] \qquad \text{and} \qquad |u(t) - w(\alpha)| \leq \delta_1\,, \ \forall\, t \in [\alpha, \alpha + \delta_2]\,.$$

We introduce seven assumptions on the nonlinear operator $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$:

(C1) Φ is a hysteresis operator;

(C2) there exists $\lambda > 0$ such that for all $\alpha \in \mathbb{R}_+$ and all $w \in \mathcal{C}_\alpha$, there exist numbers $\delta_1, \delta_2 > 0$ such that for all $u, v \in \mathcal{C}(w; \delta_1, \delta_2)$

$$\sup_{t \in [\alpha, \alpha+\delta_2]} |(\Phi(u))(t) - (\Phi(v))(t)| \leq \lambda \sup_{t \in [\alpha, \alpha+\delta_2]} |u(t) - v(t)|\,;$$

(C3) $\Phi(AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}) \subset AC(\mathbb{R}_+, \mathbb{R})$;

(C4) Φ is monotone in the sense that for all $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$ with $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$,

$$\frac{d}{dt}(\Phi(u))(t)\, \dot{u}(t) \geq 0\,, \quad \text{a.e. } t \in \mathbb{R}_+\,;$$

(C5) if $u \in \mathcal{C}$ is approximately ultimately non-decreasing and $\lim_{t\to\infty} u(t) = \infty$, then $(\Phi(u))(t)$ and $(\Phi(-u))(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$;

(C6) if, for $u \in \mathcal{C}$, $L := \lim_{t\to\infty}(\Phi(u))(t)$ exists with $L \in \operatorname{int} \mathrm{NVS}\,\Phi$, then u is bounded.

If $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$, then we introduce an additional assumption:

(C7) for all $a > 0$ and all $u \in C([0, a), \mathbb{R})$, there exist $\alpha, \beta > 0$ such that

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \leq \alpha + \beta \sup_{t\in[0,\tau]} |u(t)|\,, \quad \forall \tau \in [0, a)\,. \tag{5.1}$$

Strictly speaking, to make sense of (5.1), we have to give meaning to $(\Phi(u))(t)$, $t \in [0, a)$, when u is a continuous function defined on a *finite* interval $[0, a)$. This is done by defining $(\Phi(u))(t) = (\Phi(\mathbf{Q}_t u))(t)$ for all $t \in [0, a)$.

Remark 5.1.1 (1) Assumptions (C1), (C2) and (C7) ensure existence and uniqueness of solutions on $\mathbb{R}_+$ to the continuous-time nonlinear closed-loop system (3.14) (see Corollary 3.2.4, noting that (C2) and (C7) and the assumptions (A1) and (A2), in Section 3.2, are identical).
(2) Occasionally we refer to (C2) as a weak Lipschitz condition and any number $l > 0$ such that (C2) holds for $\lambda = l$, we call a *weak Lipschitz constant* of Φ.
(3) If (C1) and (C5) hold, then the numerical value set of Φ, $\mathrm{NVS}\,\Phi$ (defined in (4.40)), is an interval. ◇

We shall show in the following section that the assumptions (C1)–(C7) are satisfied by a large class of hysteresis operators. Some of the implications of the assumptions (C1)–(C3) are described in the following lemma.

Lemma 5.1.2 *Let $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$. For an operator $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ the following statements hold:*

(1) if Φ satisfies (C1) and (C2), then for all $u \in \mathcal{C}$ and all $\alpha \in \mathbb{R}_+$, there exists $\delta > 0$ such that for all $t \in [\alpha, \alpha + \delta]$

$$|(\Phi(u))(t) - (\Phi(u))(\alpha)| \leq \lambda \sup_{\tau\in[\alpha,t]} |u(\tau) - u(\alpha)|\,; \tag{5.2}$$

(2) if Φ satisfies (C1)-(C3), then for all $u \in AC(\mathbb{R}_+,\mathbb{R}) \cap \mathcal{C}$

$$\left|\frac{d}{dt}(\Phi(u))(t)\right| \leq \lambda|\dot{u}(t)|\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E_u\,, \tag{5.3}$$

where λ is a weak Lipschitz constant of Φ and E_u is the set of all $t \in \mathbb{R}_+$ such that u or $\Phi(u)$ is not differentiable at t.

Proof: To prove statement (1), let $u \in \mathcal{C}$ and $\alpha \in \mathbb{R}_+$ and define $w \in \mathcal{C}_\alpha$ by $w(t) = u(t)$ for all $t \in [0,\alpha]$. By (C2), there exist numbers $\delta_1, \delta_2 > 0$ such that for all $v_1, v_2 \in \mathcal{C}(w;\delta_1,\delta_2)$

$$\sup_{t\in[\alpha,\alpha+\delta_2]} |(\Phi(v_1))(t) - (\Phi(v_2))(t)| \leq \lambda \sup_{t\in[\alpha,\alpha+\delta_2]} |v_1(t) - v_2(t)|\,.$$

By continuity of u, there exists $\delta \in (0,\delta_2)$ such that $\mathbf{Q}_t u \in \mathcal{C}(w;\delta_1,\delta_2)$ for all $t \in [\alpha, \alpha+\delta]$. Thus, using, we may conclude that for $t \in [\alpha,\alpha+\delta]$

$$\begin{aligned}
|(\Phi(u))(t) - (\Phi(u))(\alpha)| &\leq \sup_{\tau\in[\alpha,t]} |(\Phi(u))(\tau) - (\Phi(u))(\alpha)| \\
&= \sup_{\tau\in[\alpha,\alpha+\delta_2]} |(\Phi(\mathbf{Q}_t u))(\tau) - (\Phi(\mathbf{Q}_\alpha u))(\tau)| \\
&\leq \lambda \sup_{\tau\in[\alpha,\alpha+\delta_2]} |(\mathbf{Q}_t u)(\tau) - (\mathbf{Q}_\alpha u)(\tau)| \\
&= \lambda \sup_{\tau\in[\alpha,t]} |u(\tau) - u(\alpha)|\,,
\end{aligned}$$

which is (5.2).

To prove statement (2), let $u \in AC(\mathbb{R}_+,\mathbb{R}) \cap \mathcal{C}$. Let E_u be the set of all $t \in \mathbb{R}_+$ such that u or $\Phi(u)$ is not differentiable at t. By (C3), E_u is of measure zero. Using statement (1), we obtain for all $t \in \mathbb{R}_+ \setminus E_u$

$$\begin{aligned}
\left|\frac{d}{dt}(\Phi(u))(t)\right| &= \lim_{\varepsilon\downarrow 0} \frac{|(\Phi(u))(t+\varepsilon) - (\Phi(u))(t)|}{\varepsilon} \\
&\leq \lambda \lim_{\varepsilon\downarrow 0} \frac{\sup_{\tau\in[t,t+\varepsilon]} |u(\tau) - u(t)|}{\varepsilon} \\
&\leq \lambda \lim_{\varepsilon\downarrow 0} \left(\sup_{\tau\in(t,t+\varepsilon]} \left|\frac{u(\tau)-u(t)}{\tau - t}\right| \right) = \lambda|\dot{u}(t)|\,,
\end{aligned}$$

which is (5.3). $\square$

Let $\Phi : \mathcal{C} \to C(\mathbb{R}_+,\mathbb{R})$ satisfy (C1)–(C4) and $u \in AC(\mathbb{R}_+,\mathbb{R})$. We define $E(\Phi,u)$ to be the set of all $t \in \mathbb{R}_+$ such that u or $\Phi(u)$ is not differentiable at t and $F(\Phi,u) := \{t \in \mathbb{R}_+ \setminus E_u \,|\, \dot{u}(t) = 0\}$. By (C3), $E(\Phi,u)$ is of measure zero. For convenience we define $G(\Phi,u) := E(\Phi,u) \cup F(\Phi,u)$.

Definition 5.1.3 Let $\mathcal{C} = C_{\rm pm}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$, $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ satisfy (C1)–(C4) and λ be a weak Lipschitz constant of Φ. Define $\Phi^\vee : AC(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ by

$$(\Phi^\vee(u))(t) = \begin{cases} \frac{d}{dt}(\Phi(u))(t)/\dot{u}(t) & \text{if } t \in \mathbb{R}_+ \setminus G(\Phi, u), \\ \lambda & \text{if } t \in G(\Phi, u). \end{cases}$$

$\diamond$

By construction, for each $u \in AC(\mathbb{R}_+, \mathbb{R})$, the function $\Phi^\vee(u)$ is measurable and by (C3), (C4) and (5.3)

$$(\Phi^\vee(u))(t) \in [0, \lambda]\,, \quad \forall\, u \in AC(\mathbb{R}_+, \mathbb{R})\,, \quad \text{a.e. } t \in \mathbb{R}_+\,.$$

By (5.3), for $t \in \mathbb{R}_+$ and $u \in AC(\mathbb{R}_+, \mathbb{R})$, we have that $\dot{u}(t) = 0$ implies $(\Phi(u))'(t) = 0$. Therefore, for $u \in AC(\mathbb{R}_+, \mathbb{R})$

$$\frac{d}{dt}(\Phi(u))(t) = (\Phi^\vee(u))(t)\dot{u}(t)\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E(\Phi, u)\,. \tag{5.4}$$

The following remark will prove useful later in the chapter (see proofs of Proposition 5.2.13 and Proposition 5.2.17).

Remark 5.1.4 Consider the following assumption which is slightly stronger than assumption (C4):

(C4′) Φ is monotone in the sense that for all $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$ with $\Phi(u) \in AC(\mathbb{R}_+, \mathbb{R})$,

$$\frac{d}{dt}(\Phi(u))(t)\, \dot{u}(t) \geq 0\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E(\Phi, u)\,.$$

If $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ satisfies (C1)–(C3) and (C4′), then,

$$(\Phi^\vee(u))(t) \in [0, \lambda]\,, \quad \forall\, u \in AC(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, t \in \mathbb{R}_+\,.$$

$\diamond$

We are now in the position to define the classes of nonlinear operators we will be considering in the context of the low-gain control problems in Chapters 6, 8 and 9.

Definition 5.1.5 Let $\lambda > 0$. The set of all operators $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfying (C1)–(C7) with $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ and having weak Lipschitz constant

λ is denoted by $\mathcal{N}_c(\lambda)$. The set of all operators $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ satisfying (C1)–(C6) with $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and having weak Lipschitz constant λ is denoted by $\mathcal{N}_{sd}(\lambda)$. $\diamond$

Remark 5.1.6 (1) The class $\mathcal{N}_c(\lambda)$ contains operators for which the continuous-time low-gain integral control results of Chapters 6 and 9 hold.
(2) The class $\mathcal{N}_{sd}(\lambda)$ contains operators for which the sample-data low-gain control results of Chapters 8 and 9 hold.
(3) (C7) is only required for the existence of solutions on $\mathbb{R}_+$ of the continuous-time closed-loop system (3.14) (see Corollary 3.2.4), but not for the sampled-data results in Chapter 8.

$\diamond$

The following lemma will be needed in Section 5.3.

Lemma 5.1.7 *Let $\Phi \in \mathcal{N}_{sd}(\lambda)$; then for every $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t_2 > t_1 \geq 0$, there exists a constant $\eta \in [0, \lambda]$ such that*

$$u \text{ affine linear on } [t_1, t_2] \quad \Longrightarrow \quad (\Phi(u))(t_2) - (\Phi(u))(t_1) = \eta(u(t_2) - u(t_1)).$$

Proof: Let $\Phi \in \mathcal{N}_{sd}(\lambda)$, $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and $t_2 > t_1 \geq 0$ and assume that u is affine linear on $[t_1, t_2]$. By (5.4),

$$(\Phi(u))(t_2) - (\Phi(u))(t_1) = \int_{t_1}^{t_2} (\Phi^{\vee}(u))(t)\dot{u}(t)\, dt. \tag{5.5}$$

Since u is affine linear on $[t_1, t_2]$, $\dot{u} \equiv (u(t_2)-u(t_1))/(t_2-t_1)$ on (t_1, t_2). Combining this with (5.5) gives

$$(\Phi(u))(t_2) - (\Phi(u))(t_1) = \frac{u(t_2) - u(t_1)}{t_2 - t_1} \int_{t_1}^{t_2} (\Phi^{\vee}(u))(t)\, dt = \eta(u(t_2) - u(t_1)),$$

where $\eta = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} (\Phi^{\vee}(u))(t)\, dt \in [0, \lambda]$. $\square$

Lemma 5.1.8 *If $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is a Lipschitz continuous hysteresis operator with Lipschitz constant $l > 0$, then assumptions (C1), (C2) (with weak Lipschitz constant $\lambda = l$), (C3) and (C7) hold.*

Proof: Let $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be Lipschitz continuous hysteresis operator with Lipschitz constant $l > 0$. Obviously (C1) holds. By causality and Lipschitz continuity (with Lipschitz constant l), it is clear that (C2) holds with weak Lipschitz constant $\lambda = l$. The fact that (C3) holds follows from Proposition 4.2.3. Finally, we show that (C7) is satisfied. To this end let $a > 0$ and $u \in C([0,a), \mathbb{R})$, then by Lipschitz continuity

$$\sup_{t\in\mathbb{R}_+} |(\Phi(\mathbf{Q}_\tau u))(t) - (\Phi(0))(t)| \leq l \sup_{t\in\mathbb{R}_+} |(\mathbf{Q}_\tau u)(t)| \,, \quad \forall \tau \in [0,a) \,.$$

Therefore, by Theorem 4.1.2, statement (1)

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \leq l \sup_{t\in[0,\tau]} |u(t)| + |(\Phi(0))(0)| \,, \quad \forall \tau \in [0,a) \,,$$

showing that (C7) is satisfied with $\alpha = (\Phi(0))(0)$ and $\beta = l$. □

Consider the following assumption which is slightly weaker than assumption (C5):

(C5′) For any ultimately non-decreasing $u \in C(\mathbb{R}_+, \mathbb{R})$ with $\lim_{t\to\infty} u(t) = \infty$

$$\lim_{t\to\infty} (\Phi(u))(t) = \sup \mathrm{NVS}\,\Phi \quad \text{and} \quad \lim_{t\to\infty} (\Phi(-u))(t) = \inf \mathrm{NVS}\,\Phi \,.$$

For future reference we state the following lemma.

Lemma 5.1.9 *Let $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be Lipschitz continuous. If Φ satisfies (C5′), then Φ satisfies (C5).*

Proof: Let $u \in C(\mathbb{R}_+, \mathbb{R})$ be approximately ultimately non-decreasing and such that $\lim_{t\to\infty} u(t) = \infty$. Then there exists a sequence of ultimately non-decreasing functions $(u_n) \subset C(\mathbb{R}_+, \mathbb{R})$ such that $u_n \stackrel{\text{uc}}{\to} u$ as $n \to \infty$. By (C5′), for all $n \in \mathbb{Z}_+$, $(\Phi(u_n))(t)$ and $(\Phi(-u_n))(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$ and therefore by Lipschitz continuity of Φ, $(\Phi(u))(t)$ and $(\Phi(-u))(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$. □

We end this section by defining the notion of critical numerical value for an operator $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$.

Definition 5.1.10 Let $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ and let $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ be an element of either $\mathcal{N}_{sd}(\lambda)$ or $\mathcal{N}_c(\lambda)$. We call $\Phi^\star \in \mathrm{NVS}\,\Phi$ a *critical numerical value* of Φ if there exists a bounded $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$, with $\lim_{t\to\infty}(\Phi(u))(t) = \Phi^\star$ and such that for all $T > 0$ and all $\varepsilon > 0$, $\mu_L\{t \geq T \,|\, (\Phi^\vee(u))(t) < \varepsilon\} > 0$. ◇

The above definition of critical numerical value might seem artificial but it is closely related to the concept of a critical value of a function, as we shall show in Section 5.2 (see subsection on static nonlinearities).

Proposition 5.1.11 *Let $\mathcal{C} = C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ or $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R})$ and let $\Phi : \mathcal{C} \to C(\mathbb{R}_+, \mathbb{R})$ be an element of either $\mathcal{N}_{sd}(\lambda)$ or $\mathcal{N}_c(\lambda)$. If $\Phi^\star \in \mathrm{NVS}\,\Phi \setminus \mathrm{int}\,(\mathrm{NVS}\,\Phi)$, then $\Phi^\star$ is a critical numerical value of Φ.*

Proof: If $\Phi^\star \in \mathrm{NVS}\,\Phi \setminus \mathrm{int}\,(\mathrm{NVS}\,\Phi) \neq \emptyset$, then there exists $v \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$ and $\tau > 0$, such that $(\Phi(v))(\tau) = \Phi^\star$. Without loss of generality we suppose that $\Phi^\star = \sup(\mathrm{NVS}\,\Phi)$. Define $u \in AC(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$ by setting $u = v$ on $[0, \tau]$ and $u(t) = v(\tau) + 1/\tau - 1/t$ for $t > \tau$. Since Φ satisfies assumptions (C3) and (C4), $(\Phi(u))(t) = \Phi^\star$ for all $t \in [\tau, \infty)$ and therefore, $\frac{d}{dt}(\Phi(u))(t) = 0$ for all $t \in (\tau, \infty)$. So $(\Phi^\vee(u))(t) = 0$ for all $t \in (\tau, \infty)$ and thus $\Phi^\star$ is a critical numerical value of Φ. □

5.2 Hysteresis operators contained in $\mathcal{N}_c(\lambda)$ and $\mathcal{N}_{sd}(\lambda)$

In this section we consider various hysteresis operators first introduced in Section 4.3, and we show that under certain extra assumptions these operators are contained in $\mathcal{N}_c(\lambda)$ (and thus in $\mathcal{N}_{sd}(\lambda)$). Additionally, for certain $\Phi \in \mathcal{N}_c(\lambda)$, we identify subsets of $\mathrm{NVS}\,\Phi$ which contain no critical numerical values of Φ.

Static nonlinearities

For a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, define the corresponding static nonlinearity

$$\mathcal{S}_\phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R}), \quad u \mapsto \phi \circ u. \tag{5.6}$$

The proof of the following proposition is straightforward and is therefore omitted.

Proposition 5.2.1 *If $\phi : \mathbb{R} \to \mathbb{R}$ is non-decreasing and globally Lipschitz with Lipschitz constant $\lambda > 0$, then the static nonlinearity $\mathcal{S}_\phi$, defined by (5.6), is contained in $\mathcal{N}_c(\lambda)$.*

The following lemma will prove useful when we consider critical numerical values of $\mathcal{S}_\phi$.

Lemma 5.2.2 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be piecewise C^1, non-decreasing and globally Lipschitz. Define $\mathcal{S}_\phi$ by (5.6). For $u \in AC(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+ \setminus G(\mathcal{S}_\phi, u)$, if $\dot{u}(t) > 0$, then $(\mathcal{S}_\phi^\vee(u))(t) = \phi'_+(u(t))$ and if $\dot{u}(t) < 0$, then $(\mathcal{S}_\phi^\vee(u))(t) = \phi'_-(u(t))$.*

Proof: Let $u \in AC(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+ \setminus G(\mathcal{S}_\phi, u)$, and suppose that $\dot{u}(t) > 0$ (the case when $\dot{u}(t) < 0$ can be treated in a similar fashion). Let $\lambda > 0$ be a Lipschitz constant of ϕ, then since $\phi'_+(u(t))$, $\dot{u}(t)$ and $(\phi \circ u)'(t)$ exist

$$
\begin{aligned}
&\left|\phi'_+(u(t))\dot{u}(t) - (\phi \circ u)'(t)\right| \\
= &\left|\lim_{h\downarrow 0} \frac{\phi(u(t) + h\dot{u}(t)) - \phi(u(t))}{h} - \lim_{h\downarrow 0} \frac{\phi(u(t+h)) - \phi(u(t))}{h}\right| \\
= &\left|\lim_{h\downarrow 0} \frac{\phi(u(t) + h\dot{u}(t)) - \phi(u(t+h))}{h}\right| \\
\leq &\ \lambda \lim_{h\downarrow 0} \left|\frac{u(t) + h\dot{u}(t) - u(t+h)}{h}\right| \\
= &\ \lambda \lim_{h\downarrow 0} \left|\frac{u(t) - u(t+h)}{h} + \dot{u}(t)\right| = 0\,,
\end{aligned}
$$

and so $(\mathcal{S}^\vee_\phi(u))(t) = \phi'_+(u(t))$. □

For a piecewise C^1 function $\phi : \mathbb{R} \to \mathbb{R}$, we call $\phi^\star \in \operatorname{im}\phi$ a *critical value* of ϕ if there exists $u \in \mathbb{R}$ such that $\phi(u) = \phi^\star$ and $\phi'_+(u)\phi'_-(u) = 0$.

Proposition 5.2.3 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be piecewise C^1, non-decreasing and globally Lipschitz. Define $\mathcal{S}_\phi$ by (5.6). Then $\Phi^\star$ is a critical numerical value of $\mathcal{S}_\phi$ if and only if $\Phi^\star$ is a critical value of ϕ.*

Proof: Suppose first that $\Phi^\star$ is a critical numerical value of $\mathcal{S}_\phi$. Then there exists a bounded $u \in AC(\mathbb{R}_+, \mathbb{R})$ such that $\lim_{t\to\infty} \phi(u(t)) = \Phi^\star$ and for all $T > 0$ and all $\varepsilon > 0$, $\mu_L\{t \geq T \,|\, (\mathcal{S}^\vee_\phi(u))(t) < \varepsilon\} > 0$. Therefore there exists $(t_n) \subset \mathbb{R}_+ \setminus G(\mathcal{S}_\phi, u)$ with $\lim_{n\to\infty} t_n = \infty$ and such that $\lim_{n\to\infty}(\mathcal{S}^\vee_\phi(u))(t_n) = 0$ and $(u(t_n))$ converges. Define $u^\infty := \lim_{n\to\infty} u(t_n)$. Choosing a subsequence if necessary, we have that either $u(t_n) > u^\infty$ for all $n \in \mathbb{Z}_+$ and $(u(t_n))$ is non-increasing, $u(t_n) < u^\infty$ for all $n \in \mathbb{Z}_+$ and $(u(t_n))$ is non-decreasing, or $u(t_n) = u_\infty$ for all $n \in \mathbb{Z}_+$. Let us first suppose that $u(t_n) > u^\infty$ for all $n \in \mathbb{Z}_+$ and $(u(t_n))$ is non-increasing (the second case can be treated in a similar fashion). There exists $N \in \mathbb{Z}_+$ such that ϕ' exists on $(u_\infty, u(t_N)]$ and therefore ϕ'_+ is continuous on $[u_\infty, u(t_N)]$. Therefore, since $\lim_{n\to\infty} \phi'_+(u(t_n)) = \lim_{n\to\infty}(\mathcal{S}^\vee_\phi(u))(t_n) = 0$, $\phi'_+(u_\infty) = 0$. Now suppose that $u(t_n) = u_\infty$ for all $n \in \mathbb{Z}_+$. Then since $\lim_{n\to\infty}(\mathcal{S}^\vee_\phi(u))(t_n) = 0$ and by Lemma 5.2.2, for each $n \in \mathbb{Z}_+$, $(\mathcal{S}^\vee_\phi(u))(t_n)$ is equal to either $\phi'_+(u(t_n))$ or $\phi'_-(u(t_n))$, we have that either $\phi'_+(u_\infty) = 0$ or $\phi'_-(u_\infty) = 0$. By continuity of ϕ, $\phi(u^\infty) = \Phi^\star$.

Now suppose that $\Phi^\star$ is a critical value of ϕ. Therefore there exists $v \in \mathbb{R}$ such that $\phi(v) = \Phi^\star$ and $\phi'_+(v)\phi'_-(v) = 0$. Let us suppose that $\phi'_+(v) = 0$ (the case when $\phi'_-(v) = 0$ can be treated in a similar fashion). Let $w > v$ be such that ϕ is continuously differentiable on $(v, w]$ and thus ϕ'_+ is continuous on $[v, w]$. Define

$u : \mathbb{R}_+ \to [v, w]$, $t \mapsto (w - v)/(1 + t) + v$. Then

$$\lim_{t\to\infty} (\mathcal{S}_\phi^\vee(u))(t) = \lim_{t\to\infty} \phi'(u(t)) = \lim_{t\to\infty} \phi'_+(u(t)) = \lim_{x\downarrow v} \phi'_+(x) = \phi'_+(v) = 0\,,$$

and thus for all $T > 0$ and all $\varepsilon > 0$, $\mu_L\{t \geq T \mid (\mathcal{S}_\phi^\vee(u))(t) < \varepsilon\} > 0$. Since, by continuity of ϕ, $\lim_{t\to\infty}(\mathcal{S}_\phi(u))(t) = \Phi^\star$, we have that $\Phi^\star$ is a critical numerical value of $\mathcal{S}_\phi$. □

Relay hysteresis

We introduced the relay hysteresis operator $\mathcal{R}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to F(\mathbb{R}_+, \mathbb{R})$ in Section 4.3. Here though we require that $\mathcal{R}_\xi$ maps into the space of continuous functions and therefore restrict or attention to "continuous" relay hysteresis operators, i.e. the two curves ρ_1 and ρ_2 join at a_1 and a_2. We note that in this case $\mathrm{NVS}\,\mathcal{R}_\xi = \mathrm{im}\,\rho_1 \cup \mathrm{im}\,\rho_2$.

Proposition 5.2.4 *If ρ_1 and ρ_2 are both non-decreasing, globally Lipschitz with Lipschitz constant $\lambda > 0$ and such that $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$, then for each $\xi \in \mathbb{R}$, the relay hysteresis operator $\mathcal{R}_\xi$, defined by (4.12), is contained in $\mathcal{N}_c(\lambda)$.*

Proof: Clearly $\mathcal{R}_\xi(C(\mathbb{R}_+, \mathbb{R})) \subset C(\mathbb{R}_+, \mathbb{R})$. A straightforward consequence of the definition of the relay hysteresis operator is that $\mathcal{R}_\xi$ satisfies conditions (C1), (C2), (C5) and (C7). To show that (C3) and (C4) hold, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. For any compact interval $J \subset \mathbb{R}_+$, u is uniformly continuous on J, and therefore, using that $a_1 \neq a_2$, there exists $\delta > 0$, such that for all $t_1, t_2 \in J$

$$u(t_1) = a_1\,,\ u(t_2) = a_2 \quad \Longrightarrow \quad |t_2 - t_1| \geq \delta\,.$$

As a consequence, there exist $0 = t_0 < t_1 < t_2 < \ldots$ with $\lim_{n\to\infty} t_n = \infty$ and a map $j : \mathbb{Z}_+ \to \{1, 2\}$ such that for all $i \in \mathbb{Z}_+$

$$(\mathcal{R}_\xi(u))(t) = \rho_{j(i)}(u(t))\,, \quad \forall\, t \in [t_i, t_{i+1}]\,. \tag{5.7}$$

It follows that $\mathcal{R}_\xi(u)$ is absolutely continuous on $[t_i, t_{i+1}]$ for each $i \in \mathbb{Z}_+$. Hence, by continuity of $\mathcal{R}_\xi(u)$, we may conclude that $\mathcal{R}_\xi(u) \in AC(\mathbb{R}_+, \mathbb{R})$, showing that (C3) holds. Furthermore, since ρ_1 and ρ_2 are non-decreasing and Lipschitz, (5.7) yields that for all $i \in \mathbb{Z}_+$

$$\frac{d}{dt}(\mathcal{R}_\xi(u))(t)\dot{u}(t) \geq 0\,, \quad \text{a.e. } t \in [t_i, t_{i+1}]\,,$$

which implies that (C4) holds.

Finally, to show that (C6) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose that $\lim_{t\to\infty}(\mathcal{R}_\xi(u))(t) = l \in \operatorname{int} \mathrm{NVS}\, \mathcal{R}_\xi$. Then there exist $\varepsilon > 0$ and $T \geq 0$ such that $I_\varepsilon := (l - \varepsilon, l + \varepsilon) \subset \operatorname{int} \mathrm{NVS}\, \mathcal{R}_\xi$ and $(\mathcal{R}_\xi(u))(t) \in I_\varepsilon$ for all $t \geq T$, which implies

$$u(t) \in \rho_1^{-1}(I_\varepsilon) \cup \rho_2^{-1}(I_\varepsilon) =: U\,, \quad \forall\, t \geq T\,. \tag{5.8}$$

But the set U is bounded since $\sup \rho_1, \inf \rho_2 \notin I_\varepsilon$ and ρ_1 and ρ_2 are non-decreasing. Combining this with (5.8) shows that u is bounded. □

The following result follows from Proposition 5.2.3.

Corollary 5.2.5 *Let ρ_1 and ρ_2 be piecewise C^1, non-decreasing, globally Lipschitz with Lipschitz constant $\lambda > 0$ and such that $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$, and let $\xi \in \mathbb{R}$. Define the relay hysteresis operator $\mathcal{R}_\xi$ by (4.12). Then $\Phi^\star$ is a critical numerical value of $\mathcal{R}_\xi$ if and only if $\Phi^\star$ is a critical value of ρ_1 or ρ_2.*†

We remark that whilst the "continuous" relay hysteresis operator $\mathcal{R}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is weakly Lipschitz continuous, $\mathcal{R}_\xi$ is not Lipschitz continuous in the sense of Definition 2.1.3. In particular, when we talk about "continuous" relay hysteresis, we simply mean that the output corresponding to a continuous input is continuous, but not that the relay hysteresis operator is continuous with respect to any natural topology on $C(\mathbb{R}_+, \mathbb{R})$.

Generalized backlash hysteresis

The generalized backlash operator $\mathcal{B}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ was introduced in Section 4.3 and shown to be a Lipschitz continuous hysteresis operator. We note that $\mathrm{NVS}\, \mathcal{B}_\xi = \operatorname{im} \beta_1 \cup \operatorname{im} \beta_2$.

Proposition 5.2.6 *Let $\xi \in \mathbb{R}$ and let $\lambda > 0$ be a Lipschitz constant of β_1 and β_2. Then the backlash operator $\mathcal{B}_\xi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is contained in $\mathcal{N}_c(\lambda)$ and additionally satisfies (C4′).*

Proof: From Proposition 4.3.2, part (3), we know that $\mathcal{B}_\xi$ is a Lipschitz continuous hysteresis operator and therefore an application of Lemma 5.1.8 implies that

†Although we have only defined a critical value for a function with domain $\mathbb{R}$, it is clear how to define a critical value for $\phi : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an interval.

(C1)–(C3) and (C7) all hold. To show that (C4′) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. We need to show that

$$\frac{d}{dt}(\mathcal{B}_\xi(u))(t)\dot{u}(t) \geq 0\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E(\mathcal{B}_\xi, u)\,. \tag{5.9}$$

Let $t \in \mathbb{R}_+ \setminus E(\mathcal{B}_\xi, u)$. If $\dot{u}(t) = 0$, then (5.9) holds trivially. If $\dot{u}(t) > 0$, then there exist $t_1 > t$ and $u_n \in C_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$ such that $\sup_{t\in[0,t_1]} |u_n(t) - u(t)| \to 0$ as $n \to \infty$ and $u_n(\tau) \geq u_n(t)$ for all $\tau \in (t, t_1)$ and all $n \in \mathbb{Z}_+$. It follows that $(\mathcal{B}_\xi(u_n))(\tau) \geq (\mathcal{B}_\xi(u_n))(t)$ for all $\tau \in (t, t_1)$, which in turn implies $(\mathcal{B}_\xi(u))(\tau) \geq (\mathcal{B}_\xi(u))(t)$ for all $\tau \in (t, t_1)$. Therefore

$$\frac{d}{dt}(\mathcal{B}_\xi(u))(t) = \lim_{\varepsilon \downarrow 0} \frac{(\mathcal{B}_\xi(u))(t+\varepsilon) - (\mathcal{B}_\xi(u))(t)}{\varepsilon} \geq 0\,,$$

and so (5.9) holds. If $\dot{u}(t) < 0$, then (5.9) can be obtained by a very similar argument.

To show that (C5) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that $(\mathcal{B}_\xi(u))(t) = \beta_1(u(t))$ for all $t \geq T$. Thus, $\lim_{t\to\infty}(\mathcal{B}_\xi(u))(t) = \sup(\operatorname{im}\beta_1)$. Similarly, $\lim_{t\to\infty}(\mathcal{B}_\xi(-u))(t) = \inf(\operatorname{im}\beta_1)$. Thus (C5′) holds and it follows from Lemma 5.1.9 that (C5) holds.

For (C6), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose $\lim_{t\to\infty}(\mathcal{B}_\xi(u))(t) = l \in \mathbb{R}$. Then there exist $\delta > 0$ and $T \in \mathbb{R}_+$ such that $(\mathcal{B}_\xi(u))(t) \in (l - \delta, l + \delta)$ for all $t \geq T$. Consequently, there exists $\varepsilon > 0$ such that $u(t) \in (\beta_1(l) - \varepsilon, \beta_2(l) + \varepsilon)$ for all $t \geq T$, and hence, u is bounded. □

Although for a given general backlash operator it is easy to say what numerical values are critical numerical values, it is none the less difficult to express this in a general proposition. We therefore, in the following result, consider only standard backlash.

Proposition 5.2.7 *Let $\mathcal{B}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be the standard backlash operator and $\Phi^\star \in \mathbb{R}$. Then $\Phi^\star$ is a critical numerical value of $\mathcal{B}_{h,\xi}$.*

Proposition 5.2.7 follows immediately from the definition of $\mathcal{B}_{h,\xi}$ and the definition of a critical numerical value.

Elastic-plastic hysteresis

To show that elastic-plastic hysteresis and the Preisach operator satisfy (C1)–(C7), we need the following lemma.

Lemma 5.2.8 *Let $u \in C(\mathbb{R}_+, \mathbb{R})$ be unbounded. Then there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} t_n = \infty$ such that either*

$$u(t_n) = \sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall\, n \in \mathbb{Z}_+ \quad \textit{or} \quad u(t_n) = -\sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

Proof: Let $u \in C(\mathbb{R}_+, \mathbb{R})$ be unbounded and $(\tau_n) \subset \mathbb{R}_+$ be an increasing sequence with $\lim_{n\to\infty} \tau_n = \infty$. By continuity of u, there exist $s_n \in [0, \tau_n]$ such that

$$|u(s_n)| = \sup_{t\in[0,\tau_n]} |u(t)| = \sup_{t\in[0,s_n]} |u(t)|\,.$$

Since u is unbounded, $\lim_{n\to\infty} \sup_{t\in[0,\tau_n]} |u(t)| = \infty$ and we can find a subsequence (t_n) of (s_n) such that either

$$u(t_n) = \sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall\, n \in \mathbb{Z}_+ \quad \text{or} \quad u(t_n) = -\sup_{t\in[0,t_n]} |u(t)|\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

□

We remark that NVS $\mathcal{E}_{h,\xi} = [-h, h]$.

Proposition 5.2.9 *For $(h, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ let $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be the elastic-plastic operator. Then:*

(1) for $H \in \mathbb{R}_+$, globally Lipschitz $\zeta : \mathbb{R}_+ \to \mathbb{R}$ with Lipschitz constant 1, $u \in C(\mathbb{R}_+, \mathbb{R})$ and $t \in \mathbb{R}_+$

$$(\mathcal{E}_{H,\zeta(H)}(u))(t) = H \quad \Longrightarrow \quad (\mathcal{E}_{h,\zeta(h)}(u))(t) = h\,, \quad \forall\, h \in [0, H]\,,$$

and

$$(\mathcal{E}_{H,\zeta(H)}(u))(t) = -H \quad \Longrightarrow \quad (\mathcal{E}_{h,\zeta(h)}(u))(t) = -h\,, \quad \forall\, h \in [0, H]\,;$$

(2) $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is contained in $\mathcal{N}_c\,(2)$ and additionally satisfies (C4′).

Proof: To prove statement (1), note that, using Lemma 4.3.3, we have for for every $u \in C(\mathbb{R}_+, \mathbb{R})$, $\xi_1, \xi_2 \in \mathbb{R}$ and $t, h_1, h_2 \in \mathbb{R}_+$

$$|(\mathcal{B}_{h_1,\xi_1}(u))(t) - (\mathcal{B}_{h_2,\xi_2}(u))(t)| \le \max(|h_1 - h_2|, |\xi_1 - \xi_2|)\,. \qquad (5.10)$$

Now let $H \in \mathbb{R}_+$, $\zeta : \mathbb{R}_+ \to \mathbb{R}$ be globally Lipschitz with Lipschitz constant 1, $u \in C(\mathbb{R}_+, \mathbb{R})$, $t \in \mathbb{R}_+$ and suppose $(\mathcal{E}_{H,\zeta(H)}(u))(t) = H$. Then using (5.10) and

Proposition 4.3.4, part (4), we have for all $h \in [0, H]$

$$H - (\mathcal{E}_{h,\zeta(h)}(u))(t) = (\mathcal{E}_{H,\zeta(H)}(u))(t) - (\mathcal{E}_{h,\zeta(h)}(u))(t) \leq H - h\,,$$

and so since $(\mathcal{E}_{h,\zeta(h)}(u))(t) \leq h$, we obtain $(\mathcal{E}_{h,\zeta(h)}(u))(t) = h$ for all $h \in [0, H]$. The second implication in statement (1) can be proved in a similar way.

To prove statement (2), we note that by Proposition 4.3.4, part (3), $\mathcal{E}_{h,\xi}$ is a Lipschitz continuous hysteresis operator and so by Lemma 5.1.8, $\mathcal{E}_{h,\xi}$ satisfies conditions (C1)–(C3) and (C7). To show (C4′) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. We need to show that

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t)\dot{u}(t) \geq 0\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E(\mathcal{E}_{h,\xi}, u)\,. \tag{5.11}$$

Let $t \in \mathbb{R}_+ \setminus E(\mathcal{E}_{h,\xi}, u)$, then by Proposition 4.3.4, part (4), u, $\mathcal{E}_{h,\xi}(u)$ and $\mathcal{B}_{h,\xi}(u)$ are all differentiable at t and

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t) = -\frac{d}{dt}(\mathcal{B}_{h,\xi}(u))(t) + \dot{u}(t)\,. \tag{5.12}$$

Since $\mathcal{B}_{h,\xi} \in \mathcal{N}_c(1)$, and $\mathcal{B}_{h,\xi}$ satisfies (C4′), it follows from Remark 5.1.4 that $(\mathcal{B}^\vee_{h,\xi}(u))(t) \in [0, 1]$ for all $t \in \mathbb{R}_+$. By (5.12)

$$\frac{d}{dt}(\mathcal{E}_{h,\xi}(u))(t) = (1 - (\mathcal{B}^\vee_{h,\xi}(u))(t))\dot{u}(t)\,, \quad \forall\, t \in \mathbb{R}_+ \setminus E(\mathcal{E}_{h,\xi}, u)\,,$$

and thus (5.11) holds.

To show that (C5) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$, then

$$\lim_{t\to\infty} (\mathcal{E}_{h,\xi}(u))(t) = h = \sup \mathrm{NVS}\, \mathcal{E}_{h,\xi}$$

and, similarly, $\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(-u))(t) = -h = \inf \mathrm{NVS}\, \mathcal{E}_{h,\xi}$. Thus (C5′) holds and it follows from Lipschitz continuity and Lemma 5.1.9 that (C5) holds.

For (C6), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose

$$\lim_{t\to\infty} (\mathcal{E}_{h,\xi}(u))(t) \in \mathrm{int}\, \mathrm{NVS}\, \mathcal{E}_{h,\xi} = (-h, h)\,.$$

Seeking a contradiction, assume that u is unbounded. Then, by Lemma 5.2.8, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) - \xi = \sup_{t\in[0,t_n]} |u(t) - \xi|$. Moreover, again without loss of generality, we may assume that $u(t_n) > h + \xi$ for all $n \in \mathbb{Z}_+$. Define for each $n \in \mathbb{Z}_+$, $H_n := u(t_n) - \xi > h$, then $(\mathcal{E}_{H_n,\xi}(u))(t_n) = H_n$ for all $n \in \mathbb{Z}_+$. By statement (1), $(\mathcal{E}_{h,\xi}(u))(t_n) = h$ for all $n \in \mathbb{Z}_+$, which is in contradiction to the assumption that $\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(u))(t) \in (-h,h)$. $\square$

To determine non-critical numerical values of the elastic-plastic and Prandtl operators we require the following lemma.

Lemma 5.2.10 *Let $(h,\xi) \in \mathbb{R}_+ \times \mathbb{R}$, $u \in AC(\mathbb{R}_+,\mathbb{R})$ and $t > 0$. Assume that $\dot{u}(t)$ exists and is non-zero. If $|(\mathcal{E}_{h,\xi}(u))(t)| = h$, then $(\mathcal{E}_{h,\xi}(u))(t) = h\,\mathrm{sign}\,(\dot{u}(t))$ and $(\mathcal{E}_{h,\xi}(u))'(t) = 0$ whenever this derivative exists. If $(\mathcal{E}_{h,\xi}(u))(t) \in (-h,h)$ then $(\mathcal{E}_{h,\xi}(u))'(t)$ exists and equals $\dot{u}(t)$.*

Proof: Let us suppose that $\dot{u}(t)$ exists and is positive (the case when $\dot{u}(t)$ is negative can be treated in a similar fashion). Then there exists $\varepsilon \in (0,t)$ such that

$$u(t-\delta) < u(t) < u(t+\delta)\,, \quad \forall\,\delta \in (0,\varepsilon)\,. \tag{5.13}$$

Let $\varepsilon_1 \in (0,\varepsilon)$ be such that $\sup_{s_1,s_2\in[t-\varepsilon_1,t+\varepsilon_1]}|u(s_1)-u(s_2)| < h$.

Suppose that $|(\mathcal{E}_{h,\xi}(u))(t)| = h$. In order to show that $(\mathcal{E}_{h,\xi}(u))(t) = h$, let us suppose, seeking a contradiction, that $(\mathcal{E}_{h,\xi}(u))(t) = -h$. Then, by the continuity of $\mathcal{E}_{h,\xi}(u)$, there exists $\varepsilon_2 \in (0,\varepsilon_1)$ such that $(\mathcal{E}_{h,\xi}(u))(\tau) < 0$ for all $\tau \in [t-\varepsilon_2,t]$. Choose $(u_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ such that $u_n(\tau) = u(\tau)$ for $\tau \in \{t-\varepsilon_2,t\}$ and all $n \in \mathbb{Z}_+$, $u_n \overset{\mathrm{uc}}{\to} u$ as $n \to \infty$,

$$\sup_{s_1,s_2\in[t-\varepsilon_2,t]}|u_n(s_1)-u_n(s_2)| < h\,, \quad \forall\, n \in \mathbb{Z}_+\,,$$

and $(\mathcal{E}_{h,\xi}(u_n))(\tau) < 0$ for all $\tau \in [t-\varepsilon_2,t]$ and all $n \in \mathbb{Z}_+$. Let $t-\varepsilon_2 = \tau_0^n < \tau_1^n < \ldots < \tau_k^n = t$ be a partition of $[t-\varepsilon_2,t]$ such that u_n is monotone on each $[\tau_i^n,\tau_{i+1}^n]$. By definition $(\mathcal{E}_{h,\xi}(u_n))(\tau_{i+1}^n) = e_h((\mathcal{E}_{h,\xi}(u_n))(\tau_i^n) + u_n(\tau_{i+1}^n) - u_n(\tau_i^n))$ and since $(\mathcal{E}_{h,\xi}(u_n))(\tau_i^n) + u_n(\tau_{i+1}^n) - u_n(\tau_i^n) < h$ we have

$$(\mathcal{E}_{h,\xi}(u_n))(\tau_{i+1}^n) \geq (\mathcal{E}_{h,\xi}(u_n))(\tau_i^n) + u_n(\tau_{i+1}^n) - u_n(\tau_i^n)\,.$$

It follows by repeated application of the above inequality, that $(\mathcal{E}_{h,\xi}(u_n))(t) \geq (\mathcal{E}_{h,\xi}(u_n))(t-\varepsilon_2) + u_n(t) - u_n(t-\varepsilon_2)$ and therefore,

$$(\mathcal{E}_{h,\xi}(u_n))(t) \geq -h + (u_n(t) - u_n(t-\varepsilon_2)) = -h + (u(t) - u(t-\varepsilon_2))\,.$$

Combining this with (5.13) and taking the limit as $n \to \infty$, we have $(\mathcal{E}_{h,\xi}(u))(t) > -h$, a contradiction. Therefore, $(\mathcal{E}_{h,\xi}(u))(t) = h$.

To show that $(\mathcal{E}_{h,\xi}(u))'(t) = 0$ whenever this derivative exists, let $(\delta_k) \subset (0,\varepsilon_1)$ be such that $\lim_{k\to\infty}\delta_k = 0$ and $u(t+\delta_k) = \max_{\tau\in[t,t+\delta_k]}u(\tau)$ for all $k \in \mathbb{Z}_+$. Let $k \in \mathbb{Z}_+$ be fixed but arbitrary. Choose $(u_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ such that $u_n(\tau) = u(t)$

for all $\tau \in [0,t]$ and all $n \in \mathbb{Z}_+$, $u_n(t+\delta_k) = u(t+\delta_k)$ for all $n \in \mathbb{Z}_+$ and $\sup_{\tau\in[t,\infty)} |u_n(\tau) - u(\tau)| \to 0$ as $n \to \infty$. Define $\nu := u(t) - h$, then for all $n \in \mathbb{Z}_+$

$$\begin{aligned}(\mathcal{E}_{h,\nu}(u_n))(t) &= (\mathcal{E}_{h,\nu}(u_n))(0) = e_h(u_n(0) - \nu) \\ &= e_h(u(t) - \nu) = e_h(h) = h = (\mathcal{E}_{h,\xi}(u))(t)\,.\end{aligned}$$

Thus

$$\lim_{n\to\infty} (\mathcal{E}_{h,\nu}(u_n))(\tau) = (\mathcal{E}_{h,\xi}(u))(\tau)\,, \quad \forall \tau \geq t\,.$$

Therefore, since $(\mathcal{E}_{h,\nu}(u_n))(t+\delta_k) = h$ for all sufficiently large $n \in \mathbb{Z}_+$, we have $(\mathcal{E}_{h,\xi}(u))(t+\delta_k) = h$. Since $k \in \mathbb{Z}_+$ was arbitrary, $(\mathcal{E}_{h,\xi}(u))(t+\delta_k) = h$ for all $k \in \mathbb{Z}_+$. Let us suppose that $(\mathcal{E}_{h,\xi}(u))'(t)$ exists, then

$$(\mathcal{E}_{h,\xi}(u))'(t) = \lim_{k\to\infty} \frac{(\mathcal{E}_{h,\xi}(u))(t+\delta_k) - (\mathcal{E}_{h,\xi}(u))(t)}{\delta_k} = 0\,.$$

If $(\mathcal{E}_{h,\xi}(u))(t) \in (-h,h)$, choose $\varepsilon_3 \in (0,\varepsilon_1)$ such that $(\mathcal{E}_{h,\xi}(u))(\tau) \in (-h,h)$ for all $\tau \in [t-\varepsilon_3, t+\varepsilon_3] =: I$. Let $(u_n) \subset C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ be such that, $u_n(\tau) = u(t-\varepsilon_3)$ for all $\tau \in [0, t-\varepsilon_3]$ and all $n \in \mathbb{Z}_+$, $u_n(\tau) = u(\tau)$ for $\tau \in \{t, t+\varepsilon_3\}$ and all $n \in \mathbb{Z}_+$, and $\sup_{\tau\in[t-\varepsilon_3,\infty)} |u_n(\tau) - u(\tau)| \to 0$ as $n \to \infty$. Define $\nu := u(t-\varepsilon_3) - (\mathcal{E}_{h,\xi}(u))(t-\varepsilon_3)$, then, for all $n \in \mathbb{Z}_+$

$$\begin{aligned}(\mathcal{E}_{h,\nu}(u_n))(t-\varepsilon_3) &= (\mathcal{E}_{h,\nu}(u_n))(0) = e_h(u_n(0) - \nu) = e_h(u(t-\varepsilon_3) - \nu) \\ &= e_h((\mathcal{E}_{h,\xi}(u))(t-\varepsilon_3)) = (\mathcal{E}_{h,\xi}(u))(t-\varepsilon_3)\,.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} (\mathcal{E}_{h,\nu}(u_n))(\tau) = (\mathcal{E}_{h,\xi}(u))(\tau)\,, \quad \forall \tau \geq t\,. \tag{5.14}$$

For all sufficiently large $n \in \mathbb{Z}_+$,

$$(\mathcal{E}_{h,\nu}(u_n))(\tau) \in (-h,h)\,, \quad \forall \tau \in I\,. \tag{5.15}$$

Let $t - \varepsilon_3 = \tau_0^n < \tau_1^n < \ldots < \tau_k^n = t + \varepsilon_3$ be a partition of $[t-\varepsilon_3, t+\varepsilon_3]$ such that u_n is monotone on each $[\tau_i^n, \tau_{i+1}^n]$. From (5.15), we know that for sufficiently large $n \in \mathbb{Z}_+$, $(\mathcal{E}_{h,\nu}(u_n))(\tau_{i+1}^n) = (\mathcal{E}_{h,\nu}(u_n))(\tau_i^n) + u_n(\tau_{i+1}^n) - u_n(\tau_i^n)$ and therefore, for sufficiently large $n \in \mathbb{Z}_+$ and all $\tau \in I$

$$(\mathcal{E}_{h,\nu}(u_n))(\tau) = u_n(\tau) - u(t-\varepsilon_3) + (\mathcal{E}_{h,\xi}(u))(t-\varepsilon_3) = u_n(\tau) - \nu\,. \tag{5.16}$$

Combining (5.14) and (5.16) we have $(\mathcal{E}_{h,\xi}(u))(\tau) = u(\tau) - \nu$ for all $\tau \in I$. Thus $(\mathcal{E}_{h,\xi}(u))'(t) = \dot{u}(t)$. □

Proposition 5.2.11 *For* $(h,\xi) \in \mathbb{R}_+ \times \mathbb{R}$ *let* $\mathcal{E}_{h,\xi} : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ *be the* *elastic-plastic operator and $\Phi^\star \in (-h,h)$. Then $\Phi^\star$ is not a critical numerical value of $\mathcal{E}_{h,\xi}$.*

Proof: From Proposition 5.2.9, part (2), we know that $\mathcal{E}_{h,\xi} \in \mathcal{N}_c(2)$. Let $\Phi^\star \in \mathrm{int}\,(\mathrm{NVS}\,\mathcal{E}_{h,\xi})$ and suppose that $u \in AC(\mathbb{R}_+,\mathbb{R})$ is bounded and such that $\lim_{t\to\infty}(\mathcal{E}_{h,\xi}(u))(t) = \Phi^\star$. Then there exists $T \in \mathbb{R}_+$ such that $(\mathcal{E}_{h,\xi}(u))(t) \in (-h,h)$ for all $t \geq T$. By Lemma 5.2.10, $(\mathcal{E}_{h,\xi}(u))'(t) = \dot{u}(t)$ for all $t \in [T,\infty) \setminus E(\mathcal{E}_{h,\xi},u)$. Then $\mathcal{E}^\vee_{h,\xi}(u)$, defined in Definition 5.1.3, is equal to 1 for all $t \in [T,\infty) \setminus G(\mathcal{E}_{h,\xi},u)$ and equal to 2 for all $t \in G(\mathcal{E}_{h,\xi},u)$ and therefore, $\mu_L\{t \geq T \,|\, (\mathcal{E}^\vee_{h,\xi}(u))(t) < 1\} = 0$ implying that $\Phi^\star$ is not a critical numerical value of $\mathcal{E}_{h,\xi}$. □

Preisach Operators

The Preisach operator $\mathcal{P}_\zeta : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ was introduced in Section 4.3 (see (4.25)). The following lemma will be useful for the verification of (C1)–(C7) for a large class of Preisach operators.

Lemma 5.2.12 *Suppose that $\mu \in \mathcal{M}_{\mathrm{lf}}(\mathbb{R}_+)$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$, $w_0 \in \mathbb{R}$, $\zeta \in \Pi$ and define $\mathcal{P}_\zeta : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ by (4.25). Let $u \in C(\mathbb{R}_+,\mathbb{R})$ and $t \in \mathbb{R}_+$. If $u(t) = \sup_{\tau\in[0,t]}|u(\tau)|$ and $\zeta = 0$ on $[u(t),\infty)$, then*

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\,ds\,d\mu(h) + w_0\,.$$

If $u(t) = -\sup_{\tau\in[0,t]}|u(\tau)|$ and $\zeta = 0$ on $[-u(t),\infty)$, then

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{-u(t)} \int_0^{u(t)+h} w(h,s)\,ds\,d\mu(h) + w_0\,.$$

Proof: Let $u \in C(\mathbb{R}_+,\mathbb{R})$, $t \in \mathbb{R}_+$ and suppose that $u(t) = \sup_{\tau\in[0,t]}|u(\tau)|$ and $\zeta = 0$ on $[u(t),\infty)$. Setting $H := u(t)$, we have $(\mathcal{B}_{H,\zeta(H)}(u))(t) = 0 = u(t) - H$ and $(\mathcal{B}_{h,\zeta(h)}(u))(t) = 0$ for all $h > H$. Combining Proposition 5.2.9, part (1) and Proposition 4.3.4, part (4), shows that $(\mathcal{B}_{h,\zeta(h)}(u))(t) = u(t) - h$ for all $h \in [0,H]$ and therefore

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\,ds\,d\mu(h) + w_0\,.$$

The second result can be proved in a similar fashion. □

Proposition 5.2.13 *Let $\mu \in \mathcal{M}_{\mathrm{lf}}(\mathbb{R}_+)$ be positive, let $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mu \otimes \mu_L)$ be non-negative and let $w_0 \in \mathbb{R}$. Suppose that $\lambda := \int_0^\infty \sup_{s\in\mathbb{R}} w(h,s)\,d\mu(h) < \infty$.*

Then, for all $\zeta \in \Pi$, the Preisach operator $\mathcal{P}_\zeta$, defined by (4.25), is contained in $\mathcal{N}_c(\lambda)$.

Remark 5.2.14 Under the assumptions of Proposition 5.2.13, we see from Lemma 5.2.12 that

$$\begin{aligned} \sup \mathrm{NVS}\, \mathcal{P}_\zeta &= \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0 \in [w_0, \infty]\,, \\ \inf \mathrm{NVS}\, \mathcal{P}_\zeta &= -\int_0^\infty \int_{-\infty}^0 w(h,s)\, ds\, d\mu(h) + w_0 \in [-\infty, w_0]\,. \end{aligned}$$

◇

Proof of Proposition 5.2.13: By Lemma 4.3.6, $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is Lipschitz continuous with Lipschitz continuity constant λ and, since $\mathcal{B}_{h,\xi}$ is a hysteresis operator, $\mathcal{P}_\zeta$ is a hysteresis operator. Therefore, by Lemma 5.1.8, $\mathcal{P}_\zeta$ satisfies conditions (C1)–(C3) and (C7).

To show that (C4) holds, let $u \in AC(\mathbb{R}_+, \mathbb{R})$. By (C3) and Lemma 4.3.6 (see also Remark 4.3.7) there exists $E \subset \mathbb{R}_+$ with $\mu_L(E) = 0$ and such that for all $t \in \mathbb{R}_+ \setminus E$, $\dot{u}(t)$ and $(\mathcal{P}_\zeta(u))'(t)$ exist, $(\mathcal{B}_{h,\zeta(h)}(u))'(t)$ exists for $|\mu|$-almost every $h \in \mathbb{R}_+$ and

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty w(h, (\mathcal{B}_{h,\zeta(h)}(u))(t))(\mathcal{B}_{h,\zeta(h)}(u))'(t)\, d\mu(h)\,. \tag{5.17}$$

Let $t \in \mathbb{R}_+ \setminus E$. If $\dot{u}(t) = 0$, (C4) immediately follows. If $\dot{u}(t) > 0$, then, since (C4$'$) holds for $\mathcal{B}_{h,\zeta(h)}$, we have $(\mathcal{B}_{h,\zeta(h)}(u))'(t) \geq 0$, whenever this derivative exists (which is the case for $|\mu|$-almost every $h \in \mathbb{R}_+$). Since w and μ are non-negative, we obtain from (5.17) that $(\mathcal{P}_\zeta(u))'(t) \geq 0$. If $\dot{u}(t) < 0$, then (C4) can be shown to hold by a similar argument.

To show that (C5) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that for all $t \geq T$, $\sup_{\tau\in[0,t]} |u(\tau)| = u(t)$ and $\zeta = 0$ on $[u(t), \infty)$. So by Lemma 5.2.12,

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} \int_0^{u(t)-h} w(h,s)\, ds\, d\mu(h) + w_0\,, \quad \forall\, t \geq T\,,$$

and since $\lim_{t\to\infty} u(t) = \infty$,

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0 \in [w_0, \infty]\,. \tag{5.18}$$

We note that because μ and w are non-negative

$$\sup \mathrm{NVS}\, \mathcal{P}_\zeta \leq \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0\,,$$

and therefore, by (5.18),

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \sup \mathrm{NVS}\, \mathcal{P}_\zeta = \int_0^\infty \int_0^\infty w(h,s)\, ds\, d\mu(h) + w_0\,. \tag{5.19}$$

Similarly, $\lim_{t\to\infty}(\mathcal{P}_\zeta(-u))(t) = \inf \mathrm{NVS}\, \mathcal{P}_\zeta$. It follows from Lipschitz continuity and Lemma 5.1.9 that (C5) holds.

For (C6), let $u \in C(\mathbb{R}_+, \mathbb{R})$ and suppose that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) \in \mathrm{int}\, \mathrm{NVS}\, \mathcal{P}_\zeta\,.$$

Let $H \in \mathbb{R}_+$ be such that $\zeta = 0$ on $[H, \infty)$. Seeking a contradiction, suppose that u is unbounded. Then, by Lemma 5.2.8, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) = \sup_{t\in[0,t_n]} |u(t)|$. Moreover, again without loss of generality, we may assume that $u(t_n) \geq H$ for all $n \in \mathbb{Z}_+$. By Lemma 5.2.12

$$(\mathcal{P}_\zeta(u))(t_n) = \int_0^{u(t_n)} \int_0^{u(t_n)-h} w(h,s)\, ds\, d\mu(h) + w_0\,, \quad \forall n \in \mathbb{Z}_+\,.$$

Since $\lim_{n\to\infty} u(t_n) = \infty$, it follows from the second equation in (5.19) that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \lim_{n\to\infty} (\mathcal{P}_\zeta(u))(t_n) = \sup \mathrm{NVS}\, \mathcal{P}_\zeta\,,$$

which is in contradiction to $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \mathrm{int}\, \mathrm{NVS}\, \mathcal{P}_\zeta$. □

The following corollary is a special case of Proposition 5.2.13.

Corollary 5.2.15 *Let μ be a finite positive Borel measure on $\mathbb{R}_+$. Then for all $\zeta \in \Pi$, the Prandtl operator $\mathcal{P}_\zeta$, defined by (4.26), is in $\mathcal{N}_c(\lambda)$, where $\lambda := \mu(\mathbb{R}_+)$.*

Remark 5.2.16 If $\mu \not\equiv 0$, then it follows from Remark 5.2.14, that for the Prandtl operator $\mathcal{P}_\zeta$, defined by (4.26), we have $\mathrm{NVS}\, \mathcal{P}_\zeta = \mathbb{R}$. ◇

An example covered by Corollary 5.2.15 is backlash hysteresis. Indeed, the backlash operator $\mathcal{B}_{h_0,\xi}$ can be obtained from (4.26) by setting $\mu = \delta_{h_0}$ (where δ_{h_0} is the unit point mass at h_0) and by letting $\zeta : \mathbb{R}_+ \to \mathbb{R}$ be any continuous function with compact support and such that $\zeta(h_0) = \xi$.

We now consider the Prandtl operator given by (4.27).

Proposition 5.2.17 *Let $p \in L^1(\mathbb{R}_+, \mathbb{R})$ be non-negative. Then for all $\zeta \in \Pi$, the Prandtl operator $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, given by (4.27), is in $\mathcal{N}_c(\lambda)$, where $\lambda := 2\int_0^\infty p(h)\,dh$.*

Remark 5.2.18 It follows from the proof below that under the assumptions of Proposition 5.2.17

$$\sup \mathrm{NVS}\,\mathcal{P}_\zeta = \int_0^\infty p(h)h\,dh \in [0,\infty], \quad \inf \mathrm{NVS}\,\mathcal{P}_\zeta = -\int_0^\infty p(h)h\,dh \in [-\infty, 0].$$

$\diamond$

Proof: By Lemma 4.3.6, $\mathcal{P}_\zeta : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ is Lipschitz continuous with Lipschitz continuity constant $\lambda = 2\int_0^\infty p(h)\,dh$ and is also a hysteresis operator and therefore by Lemma 5.1.8, satisfies conditions (C1)–(C3) and (C7).

To show that (C4) holds, fix $u \in AC(\mathbb{R}_+, \mathbb{R})$. By (C3), Proposition 4.3.4, part (4) and Lemma 4.3.6 (see also Remark 4.3.7) there exists $E \subset \mathbb{R}_+$ with $\mu_L(E) = 0$ and such that for all $t \in \mathbb{R}_+ \setminus E$, $\dot{u}(t)$ and $(\mathcal{P}_\zeta(u))'(t)$ exist, $(\mathcal{E}_{h,\zeta(h)}(u))'(t)$ exists for almost every $h \in \mathbb{R}_+$ and

$$(\mathcal{P}_\zeta(u))'(t) = \int_0^\infty p(h)(\mathcal{E}_{h,\zeta(h)}(u))'(t)\,dh.$$

Let $t \in \mathbb{R}_+ \setminus E$. If $\dot{u}(t) = 0$, (C4) immediately follows. If $\dot{u}(t) > 0$, then, since (C4$'$) holds for $\mathcal{E}_{h,\zeta(h)}$, we have $(\mathcal{E}_{h,\zeta(h)}(u))'(t) \geq 0$ whenever this derivative exists (which is the case for almost every $h \in \mathbb{R}_+$). Since p is non-negative, we may conclude that $(\mathcal{P}_\zeta(u))'(t) \geq 0$. If $\dot{u}(t) < 0$, then (C4) can be shown to hold by a similar argument.

To prove that (C5) is satisfied, let $u \in C(\mathbb{R}_+, \mathbb{R})$ be ultimately non-decreasing with $\lim_{t\to\infty} u(t) = \infty$. Then there exists $T \in \mathbb{R}_+$ such that for all $t \geq T$, $\sup_{\tau\in[0,t]} |u(\tau)| = u(t)$ and $\zeta = 0$ on $[u(t), \infty)$. So, by Lemma 5.2.12, with $w_0 = 0$, $w \equiv 1$ and $\mu = \left(\int_0^\infty p(h)\,dh\right)\delta_0 - p\mu_L$, we obtain

$$(\mathcal{P}_\zeta(u))(t) = \int_0^{u(t)} p(h)h\,dh + u(t)\int_{u(t)}^\infty p(h)\,dh, \quad \forall\, t \geq T. \tag{5.20}$$

We note that because p is non-negative, $\sup \mathrm{NVS}\,\mathcal{P}_\zeta \leq \int_0^\infty p(h)h\,dh \in [0,\infty]$. Now using (5.20) and the fact that p is non-negative

$$\int_0^\infty p(h)h\,dh \geq \sup \mathrm{NVS}\,\mathcal{P}_\zeta \geq (\mathcal{P}_\zeta(u))(t) \geq \int_0^{u(t)} p(h)h\,dh, \quad \forall\, t \geq T.$$

Since $\lim_{t\to\infty} u(t) = \infty$, it follows that

$$\lim_{t\to\infty} (\mathcal{P}_\zeta(u))(t) = \int_0^\infty p(h)h\,dh = \sup \mathrm{NVS}\,\mathcal{P}_\zeta\,.$$

Similarly, $\lim_{t\to\infty}(\mathcal{P}_\zeta(-u))(t) = \inf \mathrm{NVS}\,\mathcal{P}_\zeta$. Consequently, (C5) follows from the Lipschitz continuity of $\mathcal{P}_\zeta$ and an application of Lemma 5.1.9.

For (C6), let $u \in C(\mathbb{R}_+,\mathbb{R})$ and suppose $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \operatorname{int}\mathrm{NVS}\,\mathcal{P}_\zeta$. Let $H \in \mathbb{R}_+$ be such that $\zeta = 0$ on $[H,\infty)$. Seeking a contradiction, suppose that u is unbounded. Then, by Lemma 5.2.8, without loss of generality, we may assume that there exists an increasing sequence $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$ and $u(t_n) = \sup_{t\in[0,t_n]} |u(t)|$. Moreover, again without loss of generality, we may assume that $u(t_n) \geq H$ for all $n \in \mathbb{Z}_+$. Then, by Lemma 5.2.12

$$(\mathcal{P}_\zeta(u))(t_n) = \int_0^{u(t_n)} p(h)h\,dh + u(t_n)\int_{u(t_n)}^\infty p(h)\,dh\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

Combining this with $\lim_{n\to\infty} u(t_n) = \infty$, we may conclude as in the proof of (C5) that $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) = \lim_{n\to\infty}(\mathcal{P}_\zeta(u))(t_n) = \sup\mathrm{NVS}\,\mathcal{P}_\zeta$, which is in contradiction to $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) \in \operatorname{int}\mathrm{NVS}\,\mathcal{P}_\zeta$. □

We end the section by showing that the Prandtl operator given by (4.27) does not have any critical numerical values in the interior of its numerical value set.

Proposition 5.2.19 *Let $p \in L^1(\mathbb{R}_+,\mathbb{R})$ be non-negative. For $\zeta \in \Pi$, let the Prandtl operator $\mathcal{P}_\zeta : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$, be given by (4.27). If $\Phi^\star \in \operatorname{int}(\mathrm{NVS}\,\mathcal{P}_\zeta)$, then $\Phi^\star$ is not a critical numerical value of $\mathcal{P}_\zeta$.*

Proof: By Proposition 5.2.17, $\mathcal{P}_\zeta \in \mathcal{N}_c(\lambda)$, where $\lambda := 2\int_0^\infty p(h)\,dh$. Let $\Phi^\star \in \operatorname{int}(\mathrm{NVS}\,\mathcal{P}_\zeta)$. Seeking a contradiction suppose that $\Phi^\star$ is a critical numerical value of $\mathcal{P}_\zeta$. Then there exists a bounded function $u \in AC(\mathbb{R}_+,\mathbb{R})$ with $\lim_{t\to\infty}(\mathcal{P}_\zeta(u))(t) = \Phi^\star$ and such that for all $T > 0$ and $\varepsilon > 0$, $\mu_L\{t \geq T \,|\, (\mathcal{P}_\zeta^\vee(u))(t) < \varepsilon\} > 0$, where $\mathcal{P}_\zeta^\vee(u)$ is given by Definition 5.1.3. In particular we can choose $(t_n) \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty}(\mathcal{P}_\zeta^\vee(u))(t_n) = 0$ and $(\mathcal{E}_{h,\zeta(h)}(u))'(t_n)$ exists for a.e. $h \in \mathbb{R}_+$ (possible by Remark 4.3.7 and Proposition 4.3.4, part(4)).

Define $H_n = \sup\{h \in \mathbb{R}_+ \,|\, |(\mathcal{E}_{h,\zeta(h)}(u))(t_n)| = h\}$. Since u is bounded, $(H_n) \subset \mathbb{R}_+$ is bounded and therefore without loss of generality we can assume that (H_n) converges. We denote the limit by $H \in \mathbb{R}_+$. Again without loss of generality we can assume that $\dot{u}(t_n) > 0$ for all $n \in \mathbb{Z}_+$. From Proposition 5.2.9, part (1), $|(\mathcal{E}_{h,\zeta(h)}(u))(t_n)| = h$ for all $h \in [0, H_n]$ and moreover, $(\mathcal{E}_{h,\zeta(h)}(u))(t_n) \in (-h,h)$ for all $h > H_n$. Thus applying Lemma 5.2.10, $(\mathcal{E}_{h,\zeta(h)}(u))(t_n) = h$ for all $h \in$

$[0, H_n]$, $(\mathcal{E}_{h,\zeta(h)}(u))'(t_n) = 0$ for all $h \in [0, H_n]$ such that this derivative exists and $(\mathcal{E}_{h,\zeta(h)}(u))'(t_n) = \dot{u}(t_n)$ for all $h > H_n$. Therefore, using Lemma 4.3.6 combined with Proposition 4.3.4, part (4), for all $n \in \mathbb{Z}_+$

$$(\mathcal{P}_\zeta^\vee(u))(t_n) = \frac{(\mathcal{P}_\zeta(u))'(t_n)}{\dot{u}(t_n)} = \int_0^\infty p(h) \frac{(\mathcal{E}_{h,\zeta(h)}(u))'(t_n)}{\dot{u}(t_n)} \, dh = \int_{H_n}^\infty p(h) \, dh \, .$$

Since $\lim_{n\to\infty}(\mathcal{P}_\zeta^\vee(u))(t_n) = 0$, $p(h) = 0$ for a.e. $h \in [H, \infty)$. Now for all $n \in \mathbb{Z}_+$

$$\begin{aligned} (\mathcal{P}_\zeta(u))(t_n) &= \int_0^H p(h)(\mathcal{E}_{h,\zeta(h)}(u))(t_n) \, dh \\ &= \int_0^{H_n} p(h) h \, dh + \int_{H_n}^H p(h)(\mathcal{E}_{h,\zeta(h)}(u))(t_n) \, dh \, . \end{aligned}$$

Therefore,

$$\Phi^\star = \lim_{n\to\infty} (\mathcal{P}_\zeta(u))(t_n) = \int_0^H p(h) h \, dh = \int_0^\infty p(h) h \, dh = \sup\,(\mathrm{NVS}\,\mathcal{P}_\zeta) \, ,$$

which is in contradiction to the fact that $\Phi^\star \in \mathrm{int}\,(\mathrm{NVS}\,\mathcal{P}_\zeta)$. □

5.3 A class of discrete-time hysteresis operators

Let $u \in F(\mathbb{Z}_+, \mathbb{R})$. The function u is called *ultimately non-decreasing* if there exists $m \in \mathbb{Z}_+$ such that u is non-decreasing on $\mathbb{Z}_+ \setminus [0, m]$.

We introduce the following four assumptions on the operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$:

(D1) Φ is a hysteresis operator;

(D2) there exists $\lambda > 0$ such that for all $u \in F(\mathbb{Z}_+, \mathbb{R})$ and all $n \in \mathbb{Z}_+$

$$u(n) \neq u(n+1) \quad \Longrightarrow \quad \frac{(\Phi(u))(n+1) - (\Phi(u))(n)}{u(n+1) - u(n)} \in [0, \lambda] \, ;$$

(D3) if $u \in F(\mathbb{Z}_+, \mathbb{R})$ is ultimately non-decreasing and $\lim_{n\to\infty} u(n) = \infty$, then $(\Phi(u))(n)$ and $(\Phi(-u))(n)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $n \to \infty$;

(D4) if, for $u \in F(\mathbb{Z}_+, \mathbb{R})$, $L := \lim_{n\to\infty}(\Phi(u))(n)$ exists with $L \in \mathrm{int}\,(\mathrm{clos}\,\mathrm{NVS}\,\Phi)$, then u is bounded.

Remark 5.3.1 (1) We note that if (D1) holds, then (D2) is implied by the monotonicity condition

$$[(\Phi(u))(n+1)-(\Phi(u))(n)][u(n+1)-u(n)] \geq 0\,, \quad \forall\, u \in F(\mathbb{Z}_+,\mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,,$$

together with the Lipschitz continuity condition

$$\sup_{n\in\mathbb{Z}_+} |(\Phi(u))(n)-(\Phi(v))(n)| \leq \lambda \sup_{n\in\mathbb{Z}_+} |u(n)-v(n)|\,, \quad \forall\, u,v \in F(\mathbb{Z}_+,\mathbb{R})\,.$$

(2) If (D1) and (D2) hold, then

$$|(\Phi(u))(n+1)-(\Phi(u))(n)| \leq \lambda\, |u(n+1)-u(n)|\,, \quad \forall\, u \in F(\mathbb{Z}_+,\mathbb{R})\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

Thus if (D3) also holds, clos (NVS Φ) is an interval. However, it can be shown that NVS Φ is not necessarily an interval, see Appendix 5 for a counterexample.

$\diamond$

If $\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ satisfies (D2), then any number $l > 0$ such that (D2) holds for $\lambda = l$, is called a *weak Lipschitz constant* of Φ.

Definition 5.3.2 Let $\Phi \in \mathcal{N}_d(\lambda)$. For $u \in F(\mathbb{Z}_+,\mathbb{R})$, define $\delta\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ by

$$(\delta\Phi(u))(n) = \begin{cases} \frac{(\Phi(u))(n+1)-(\Phi(u))(n)}{u(n+1)-u(n)} & \text{if } u(n+1) \neq u(n), \\ \lambda & \text{if } u(n+1) = u(n). \end{cases}$$

$\diamond$

By construction, using (D2), $(\delta\Phi(u))(n) \in [0,\lambda]$ for all $n \in \mathbb{Z}_+$ and all $u \in F(\mathbb{Z}_+,\mathbb{R})$. By (D1) we have that for all $u \in F(\mathbb{Z}_+,\mathbb{R})$

$$(\Phi(u))(n+1)-(\Phi(u))(n) = (\delta\Phi(u))(n)(u(n+1)-u(n))\,, \quad \forall\, n \in \mathbb{Z}_+\,.$$

We are now in a position to define the class of nonlinear operators we will be considering in the context of the discrete-time integral control problem in Chapter 7.

Definition 5.3.3 Let $\lambda > 0$. The set of all operators $\Phi : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ satisfying (D1)–(D4) and having weak Lipschitz constant λ is denoted by $\mathcal{N}_d(\lambda)$.

$\diamond$

We now consider the discrete-time backlash operator and show that it satisfies (D1)–(D4).

Example 5.3.4 Let $h \in \mathbb{R}_+$ be arbitrary. Define the function $b_h : \mathbb{R}^2 \to \mathbb{R}$ by (4.20). We note that

$$b_h(v,w) \in [v-h, v+h], \quad \forall\, v,w \in \mathbb{R}, \tag{5.21}$$

$$b_h(v,w) = w, \quad \forall\, (v,w) \in \{(z_1,z_2) \,|\, z_1 \in \mathbb{R}, z_2 \in [z_1 - h, z_1 + h]\}, \tag{5.22}$$

$$(b_h(v_1,w) - b_h(v_2,w))(v_1 - v_2) \geq 0, \quad \forall\, v_1, v_2, w \in \mathbb{R}. \tag{5.23}$$

Let $\mathcal{B}^d_{h,\xi} : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ be the discrete-time backlash operator defined in Example 4.5.7. We show that $\mathcal{B}^d_{h,\xi} \in \mathcal{N}_d(1)$. By Proposition 4.5.4 we know that $\mathcal{B}^d_{h,\xi}$ satisfies (D1) (since $\mathcal{B}^d_{h,\xi}$ is the discretization of the continuous-time hysteresis operator $\mathcal{B}_{h,\xi}$). Combining (5.21)–(5.23) leads to

$$[(\mathcal{B}^d_{h,\xi}(u))(n+1) - (\mathcal{B}^d_{h,\xi}(u))(n)][u(n+1) - u(n)] \;\geq\; 0,$$
$$\forall\, u \in F(\mathbb{Z}_+,\mathbb{R}), \quad \forall\, n \in \mathbb{Z}_+. \tag{5.24}$$

From Lemma 4.3.3

$$|b_h(v_1,w_1) - b_h(v_2,w_2)| \leq \max(|v_1 - v_2|, |w_1 - w_2|), \quad \forall\, v_1, v_2, w_1, w_2 \in \mathbb{R}.$$

Thus,

$$|(\mathcal{B}^d_{h,\xi}(u))(n+1) - (\mathcal{B}^d_{h,\xi}(u))(n)| \leq |u(n+1) - u(n)|, \quad \forall\, u \in F(\mathbb{Z}_+,\mathbb{R}), \quad \forall\, n \in \mathbb{Z}_+,$$

which combined with (5.24) implies that (D2) holds for $\lambda = 1$. Note that $\mathrm{NVS}\,\mathcal{B}^d_{h,\xi} = \mathbb{R}$. By (5.21), for all $u \in F(\mathbb{Z}_+,\mathbb{R})$ and all $n \in \mathbb{Z}_+$, $(\mathcal{B}^d_{h,\xi}(u))(n) \in [u(n) - h, u(n) + h]$, showing that (D3) holds. Finally, it is clear that

$$v \in [b_h(v,w) - h, b_h(v,w) + h], \quad \forall\, v,w \in \mathbb{R},$$

and so

$$u(n) \in [(\mathcal{B}^d_{h,\xi}(u))(n) - h, (\mathcal{B}^d_{h,\xi}(u))(n) + h], \quad \forall\, u \in F(\mathbb{Z}_+,\mathbb{R}), \quad \forall\, n \in \mathbb{Z}_+,$$

showing that (D4) is satisfied. We have shown that (D1)–(D4) hold for $\mathcal{B}^d_{h,\xi}$ (with $\lambda = 1$) and hence $\mathcal{B}^d_{h,\xi} \in \mathcal{N}_d(1)$. ◇

In fact what we have shown above directly is true in general, that is the discretization of an element of $\mathcal{N}_{sd}(\lambda)$ is an element of $\mathcal{N}_d(\lambda)$. This is expressed in the following proposition which is the main result of this section.

Proposition 5.3.5 *Let $\Phi \in \mathcal{N}_{sd}(\lambda)$ and let $\Phi^d : F(\mathbb{Z}_+,\mathbb{R}) \to F(\mathbb{Z}_+,\mathbb{R})$ be defined by (4.39). Then $\Phi^d \in \mathcal{N}_d(\lambda)$.*

Proof: By Proposition 4.5.4, (D1) holds. Let $u \in F(\mathbb{Z}_+, \mathbb{R})$, $n \in \mathbb{Z}_+$ and suppose that $u(n+1) \neq u(n)$. Then by Proposition 4.5.6 and Lemma 5.1.7 there exists a constant $\eta \in [0, \lambda]$ such that

$$\begin{aligned}(\Phi^d(u))(n+1) - (\Phi^d(u))(n) &= (\Phi(P_\tau u))((n+1)\tau) - (\Phi(P_\tau u))(n\tau) \\ &= \eta[(P_\tau u)((n+1)\tau) - (P_\tau u)(n\tau)] \\ &= \eta[u(n+1) - u(n)],\end{aligned}$$

and thus (D2) holds. Since $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$ (by Proposition 4.5.6) and Φ satisfies (C5), (D3) follows from an application of Proposition 4.5.6. Finally, to show that (D4) is satisfied, let $u \in F(\mathbb{Z}_+, \mathbb{R})$ be such that $\lim_{n\to\infty}(\Phi^d(u))(n)$ exists and

$$L := \lim_{n\to\infty} (\Phi^d(u))(n) \in \mathrm{int}\,(\mathrm{clos}\,(\mathrm{NVS}\,\Phi^d)). \tag{5.25}$$

By Proposition 4.5.6,

$$\lim_{n\to\infty} (\Phi(P_\tau u))(n\tau) = L. \tag{5.26}$$

Clearly $P_\tau u$ is monotone on $[n\tau, (n+1)\tau]$ for each $n \in \mathbb{Z}_+$ and therefore, by the fact that Φ satisfies (C4), $\Phi(P_\tau u)$ is monotone on $[n\tau, (n+1)\tau]$ for each $n \in \mathbb{Z}_+$. Combining this with (5.26) shows that

$$\lim_{t\to\infty} (\Phi(P_\tau u))(t) = L.$$

By Remark 5.2.14, $\mathrm{NVS}\,\Phi$ is an interval; since by Proposition 4.5.6, $\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi$, it follows from (5.25) that $L \in \mathrm{int}\,(\mathrm{clos}\,(\mathrm{NVS}\,\Phi))$. Now Φ satisfies (C6), and so we may conclude that $P_\tau u$ and thus u are bounded. $\square$

As in continuous-time, we introduce a concept of critical numerical value of an operator $\Phi \in \mathcal{N}_d(\lambda)$.

Definition 5.3.6 We call $\Phi^\star \in \mathrm{int}\,(\mathrm{clos}\,(\mathrm{NVS}\,\Phi^{\mathrm{d}}))$ a *critical numerical value* of $\Phi \in \mathcal{N}_d(\lambda)$ if there exists a bounded $u \in F(\mathbb{Z}_+, \mathbb{R})$, with $\lim_{n\to\infty}(\Phi(u))(n) = \Phi^\star$ and

$$\liminf_{n\to\infty} (\delta\Phi(u))(n) = 0.$$

$\diamond$

Proposition 5.3.7 *Let $\Phi \in \mathcal{N}_{sd}(\lambda)$ and let $\Phi^d : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ be defined by (4.39). If $\Phi^\star$ is a critical numerical value of Φ^d, then $\Phi^\star$ is a critical numerical value of Φ.*

Proof: By Proposition 4.5.6, $\mathrm{NVS}\,\Phi = \mathrm{NVS}\,\Phi^d$. Let $\Phi^\star$ be a critical numerical value of Φ^d. Then there exists a bounded $u \in F(\mathbb{Z}_+, \mathbb{R})$ such that

$\lim_{n\to\infty}(\Phi^d(u))(n) = \Phi^\star$ and $\liminf_{n\to\infty}(\delta\Phi^d(u))(n) = 0$. Defining $v := P_\tau u \in AC(\mathbb{R}_+,\mathbb{R}) \cap C_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$, $\Phi(v)$ is monotone on each $[n\tau,(n+1)\tau]$. Therefore, by (4.41), $\lim_{t\to\infty}(\Phi(v))(t) = \Phi^\star$. It is clear that

$$\tau(\delta\Phi^d(u))(n) = \int_{n\tau}^{(n+1)\tau} (\Phi^\vee(v))(t)\,dt\,,$$

and therefore since $\liminf_{n\to\infty}(\delta\Phi^d(u))(n) = 0$, for all $T > 0$ and all $\varepsilon > 0$, $\mu_L\{t \geq T \mid (\Phi^\vee(v))(t) < \varepsilon\} > 0$. Thus $\Phi^\star$ is a critical numerical value of Φ. $\square$

5.4 Notes and references

The assumptions (C1)–(C7) were first introduced by Logemann and Mawby in [19] (see (N1)–(N8) in [19]) and later appeared in [21]. The majority of Section 5.2 is also contained in [19] by Logemann and Mawby. The assumptions (D1)–(D4) were first introduced by Logemann and Mawby in [21]. All the results of Section 5.1 are new. The concept of a critical numerical value is seen here for the first time. The results of Section 5.2, which show that specific operators are contained in $\mathcal{N}_c(\lambda)$, and those dealing with critical numerical values, are new. Proposition 5.2.9, part (1), is a property of the elastic-plastic hysteresis operator which could not be located in the literature, and Lemma 5.2.10, although contained in [4] (see Lemma 2.3.8 in [4]), was not proved there. Section 5.3 consists entirely of new material.

Chapter 6

Low-gain integral control of continuous-time regular linear systems subject to input hysteresis

6.1 Integral control in the presence of input nonlinearities in $\mathcal{N}_c(\lambda)$

We consider the following nonlinear system of differential equations (see Figure 2)

$$\dot{x} = Ax + B\Phi(u)\,, \quad x(0) = x_0 \in X\,, \tag{6.1a}$$

$$\dot{u} = k[r - C_L x - D\Phi(u)]\,, \quad u(0) = u_0 \in \mathbb{R}\,, \tag{6.1b}$$

where k is a real parameter, $(A, B, C, D) \in \mathcal{L}$ and $\Phi \in \mathcal{N}_c(\lambda)$. We recall that for $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R}, \quad t \mapsto (x(t), u(t))$$

is called a *solution* of (6.1) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R})$-valued function, $x(t) \in \operatorname{dom}(C_L)$ for a.e. $t \in [0, a)$, $(x(0), u(0)) = (x_0, u_0)$ and the differential equations in (6.1) are satisfied almost everywhere on $[0, a)$, where the derivative in (6.1a) should be interpreted in the space X_{-1}.†

† Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [2], Theorem 3.1, p. 10.

The next result asserts that (6.1) has a unique solution on the whole of $\mathbb{R}_+$.

Lemma 6.1.1 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$ and $r \in \mathbb{R}$. For each $(x_0, u_0) \in X \times \mathbb{R}$, there exists a unique solution $(x(\cdot), u(\cdot))$ of (6.1) defined on $\mathbb{R}_+$.*

Proof: To recover (6.1) from (3.14), set $h \equiv 0$, $\theta_0 = 1$ and $\kappa \equiv k$. Then the result follows from Corollary 3.2.4. □

Let $\mathbf{G}(s)$ be the transfer function of (A, B, C, D). If $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha)$ for some $\alpha < 0$ (which is the case if $\mathbf{T}$ is exponentially stable) and $\mathbf{G}(0) > 0$, then it is easy to show that

$$1 + k \operatorname{Re} \frac{\mathbf{G}(s)}{s} \geq 0\,, \quad \forall\, s \in \mathbb{C}_0\,, \tag{6.2}$$

for all sufficiently small $k > 0$, see Lemma 3.10 in [28]. We define

$$K := \sup\{k > 0 \,|\, (6.2) \text{ holds}\}\,. \tag{6.3}$$

Henceforth, let $\mathcal{M}_{\mathrm{f}}(\mathbb{R}_+) \subset \mathcal{M}(\mathbb{R}_+)$ denote the space of all finite signed Borel measures on $\mathbb{R}_+$. Recall that a signed measure μ on $\mathbb{R}_+$ is called finite if $|\mu|(\mathbb{R}_+) < \infty$. For $\alpha \in \mathbb{R}$, we define the exponentially weighted space $\mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ as the set of all $\mu \in \mathcal{M}_{\mathrm{lf}}(\mathbb{R}_+)$ with the property that the weighted measure $E \mapsto \int_E e^{-\alpha t}\, d\mu(t)$ belongs to $\mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$.

The main result of this chapter is the following theorem.

Theorem 6.1.2 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$, $k \in (0, K/\lambda)$ and $r \in \mathbb{R}$ is such that*

$$\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\operatorname{NVS} \Phi)\,. \tag{6.4}$$

Then, we have that for all $(x_0, u_0) \in X \times \mathbb{R}$, a unique solution $(x(\cdot), u(\cdot))$ of (6.1) exists on $\mathbb{R}_+$ and satisfies

(1) $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r\,,$

(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\Phi_r\| = 0\,,$

(3) $\lim_{t\to\infty}[r - y(t) + (\Psi_\infty x_0)(t)] = 0\,,$ *where* $y(t) = C_L x(t) + D(\Phi(u))(t)\,,$

(4) *if* $\Phi_r \in \operatorname{int}(\operatorname{NVS}\Phi)$, *then* $u(\cdot)$ *is bounded,*

(5) *if* $\Phi_r \in \operatorname{int}(\operatorname{NVS}\Phi)$ *and* Φ_r *is not a critical numerical value of* Φ, *then the convergence in (1) and (2) is of order* $\exp(-\rho t)$ *for some* $\rho > 0$; *moreover, if* $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ *for some* $\alpha < 0$, *then the convergence in (3) is of order,* $\exp(-\rho t)$ *for some* $\rho \in (0, -\alpha)\,,$

(6) *if* $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ *and* Φ_r *is not a critical numerical value of* Φ*, then there exists* $u_\infty \in \mathbb{R}$ *such that* $\lim_{t\to\infty} u(t) = u_\infty$.

Remark 6.1.3 (1) Since $(\Psi_\infty x_0)(t)$ converges exponentially to 0 as $t \to \infty$ for all $x_0 \in X_1 = \mathrm{dom}(A)$, it follows from (3) that the error $e(t) = r - y(t)$ converges to 0 for all $x_0 \in \mathrm{dom}(A)$. If C is bounded, then this statement is true for all $x_0 \in X$. If C is unbounded and $x_0 \notin \mathrm{dom}(A)$, then $e(t)$ does not necessarily converge to 0 as $t \to \infty$. However, the proof of Theorem 6.1.2 will show that $e(t)$ is small for large t in the sense that $e(t) = e_1(t) + e_2(t)$, where the function e_1 is bounded with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$.

(2) The assumption that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$ (or that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}^\alpha_{\mathrm{f}}(\mathbb{R}_+)$ for some $\alpha < 0$) is not very restrictive and seems to be satisfied in all practical examples of exponentially stable regular systems. In particular, this assumption is satisfied if B or C is bounded (see Lemma 2.3 in [24]).

(3) In applying Theorem 6.1.2 it is important to know the constant K or at least a lower bound for K. In principle, K can be obtained from frequency/step response experiments performed on the linear part of the plant, see [25] for details. $\diamond$

Proof of Theorem 6.1.2: By Lemma 6.1.1, there exists a unique solution of (6.1) on $\mathbb{R}_+$. We denote this solution by $(x(\cdot), u(\cdot))$ and introduce new variables by defining

$$z(t) := x(t) + A^{-1}B(\Phi(u))(t)\,, \quad v(t) := (\Phi(u))(t) - \Phi_r\,; \quad \forall\, t \geq 0\,.$$

By regularity it follows that $z(t) \in \mathrm{dom}\,(C_L)$ for a.e. $t \in \mathbb{R}_+$. For convenience we let $d_u := \Phi^\vee(u)$ (recall $\Phi^\vee$ from Definition 5.1.3). Then, by (5.4), $\frac{d}{dt}(\Phi(u))(t) = d_u(t)\dot{u}(t)$ for a.e. $t \in \mathbb{R}_+$. Therefore an easy calculation yields that for a.e. $t \in \mathbb{R}_+$

$$\dot{z}(t) = Az(t) + A^{-1}Bw(t)\,, \quad z(0) = z_0\,, \tag{6.5a}$$

$$\dot{v}(t) = w(t)\,, \quad v(0) = v_0\,, \tag{6.5b}$$

where

$$w(t) = -kd_u(t)(C_L z(t) + \mathbf{G}(0)v(t))\,,$$

and

$$z_0 := x_0 + A^{-1}B(\Phi(u))(0)\,, \quad v_0 := (\Phi(u))(0) - \Phi_r\,.$$

The derivative on the left-hand side of (6.5a) has to be understood in X_{-1}. Choose $c \in (k\lambda, K)$. We consider the quadruple $\Xi = (A, A^{-1}B, C, 1/c)$ of operators, which are the generating operators of an exponentially stable Pritchard-Salamon system† on the spaces $X_1 \hookrightarrow X$ (this implies that Ξ defines an exponentially stable regular system). The function

$$\mathbf{H}(s) = C(sI - A)^{-1}A^{-1}B + 1/c,$$

is the transfer function of Ξ. Then

$$\mathbf{H}(s) = \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0)) + 1/c.$$

By the definition of K there exists $\delta > 0$ such that

$$\operatorname{Re}\mathbf{H}(i\omega) \geq \delta, \quad \forall \omega \in \mathbb{R}.$$

Consequently, by a result in van Keulen [15] (see Theorem 3.10 and Remark 3.14 in [15]), there exists $\tilde{P} \in L(X)$, $\tilde{P} = \tilde{P}^*$, such that

$$\begin{aligned}\langle Ax_1, \tilde{P}x_2\rangle + \langle \tilde{P}x_1, Ax_2\rangle \ &= \\ &\frac{c}{2}\langle [(A^{-1}B)^*\tilde{P} + C]x_1, [(A^{-1}B)^*\tilde{P} + C]x_2\rangle, \quad \forall x_1, x_2 \in X_1. \quad (6.6)\end{aligned}$$

Setting

$$P := -\tilde{P} \in L(X), \qquad M := \sqrt{\frac{c}{2}}[C - (A^{-1}B)^*P] \in L(X_1, \mathbb{R}),$$

we obtain, using (6.6),

$$\begin{aligned}\langle Ax_1, Px_2\rangle + \langle Px_1, Ax_2\rangle &= -(Mx_1)(Mx_2), \quad \forall x_1, x_2 \in X_1, & (6.7a)\\ (A^{-1}B)^*Px &= Cx - \sqrt{2/c}Mx, \quad \forall x \in X_1. & (6.7b)\end{aligned}$$

We show that $P \geq 0$. Let $z_0 \in X_1$ and define $z(t) = \mathbf{T}_t z_0$. Then using (6.7a),

$$\frac{d}{dt}\langle z(t), Pz(t)\rangle = \langle Az(t), Pz(t)\rangle + \langle Pz(t), Az(t)\rangle = -(Mz(t))^2,$$

for all $t > 0$. Since $\mathbf{T}$ is exponentially stable we can integrate the above from 0 to ∞ to obtain $-\langle z_0, Pz_0\rangle = -\int_0^\infty (Mz(t))^2\,dt \leq 0$. Therefore $\langle z_0, Pz_0\rangle \geq 0$ for all $z_0 \in X_1$. Since X_1 is dense in X we can infer that $P \geq 0$.

For an intermediate step in the stability analysis we need differentiability in X, and therefore we will use an approximation argument. To this end let $T > 0$ be fixed, but arbitrary, and choose $(z_0^n) \subset X_1$ such that

$$\lim_{n\to\infty} \|z_0 - z_0^n\| = 0. \qquad (6.8)$$

†See [15] for the concept of a Pritchard-Salamon system.

Consider the system

$$\dot{\eta}(t) = A\eta(t) + A^{-1}Bw(t) , \quad \eta(0) = z_0^n . \tag{6.9}$$

The abstract initial-value problem (6.9) has a strong solution z_n on $[0,T]$ in the sense that $z_n(0) = z_0^n$ and (6.9) is satisfied for a.e. $t \in [0,T]$ in X (see Pazy [34], Theorem 2.9, p. 109). Using (6.8) we obtain

$$\lim_{n\to\infty} \|z - z_n\|_{L^2([0,T],X)} = 0 ; \qquad \lim_{n\to\infty} \|z(t) - z_n(t)\| = 0 , \quad \forall\, t \in [0,T] .$$

By (6.7b), M is an admissible output operator for $\mathbf{T}$ with $\operatorname{dom} M = \operatorname{dom} C$. Moreover, defining M_L by (3.4) with C and C_L replaced by M and M_L, respectively, it follows from (6.7b) that

$$M_L = \sqrt{c/2}(C_L - (A_{-1}B)^* P)$$

with $\operatorname{dom} M_L = \operatorname{dom} C_L$. Since for a.e. $t \in [0,T]$

$$C_L z(t) - Cz_n(t) = C_L\mathbf{T}_t z_0 - C\mathbf{T}_t z_0^n , \quad M_L z(t) - Mz_n(t) = M_L\mathbf{T}_t z_0 - M\mathbf{T}_t z_0^n ,$$

we have

$$\lim_{n\to\infty} \|C_L z - Cz_n\|_{L^2([0,T],\mathbb{R})} = 0 ; \qquad \lim_{n\to\infty} \|M_L z - Mz_n\|_{L^2([0,T],\mathbb{R})} = 0 . \tag{6.10}$$

Differentiating the function

$$\tau \mapsto V_n(\tau) = \langle z_n(\tau), Pz_n(\tau)\rangle + \mathbf{G}(0)v(\tau)^2 ,$$

we obtain for a.e. $\tau \in [0,T]$,

$$\begin{aligned} \dot{V}_n(\tau) &= \langle Az_n(\tau), Pz_n(\tau)\rangle + \langle Pz_n(\tau), Az_n(\tau)\rangle \\ &\quad +2w(\tau)(A^{-1}B)^* Pz_n(\tau) + 2\mathbf{G}(0)v(\tau)w(\tau) . \end{aligned}$$

Since $z_n(\tau) \in X_1$ for all $\tau \in [0,T]$, we may use (6.7) to obtain

$$\dot{V}_n(\tau) = -(Mz_n(\tau))^2 + 2w(\tau)(C_L z_n(\tau) - \sqrt{2/c}Mz_n(\tau)) + 2\mathbf{G}(0)v(\tau)w(\tau) .$$

For $0 \le s < t \le T$, integration from s to t gives

$$V_n(t) - V_n(s) = \int_s^t \left[-(Mz_n)^2 + 2w(C_L z_n - \sqrt{2/c}Mz_n) + 2\mathbf{G}(0)vw\right] . \tag{6.11}$$

Then taking limits in (6.11) as $n \to \infty$, invoking (6.10) and setting

$$V(\tau) = \langle z(\tau), Pz(\tau)\rangle + \mathbf{G}(0)v(\tau)^2\,, \tag{6.12}$$

we obtain

$$V(t) - V(s) = \int_s^t \left[-(M_L z)^2 + 2w(C_L z - \sqrt{2/c}M_L z) + 2\mathbf{G}(0)vw\right]\,.$$

Completing the square gives

$$\begin{aligned} V(t) - V(s) \quad &= \quad -\int_s^t (M_L z - kd_u\sqrt{2/c}(C_L z + \mathbf{G}(0)v))^2 \\ &\quad + \int_s^t (k^2 d_u^2 2/c - 2kd_u)(C_L z + \mathbf{G}(0)v)^2\,, \end{aligned} \tag{6.13}$$

which holds for all $s, t \in [0, T]$ with $s < t$. Since $T > 0$ was arbitrary, it follows that (6.13) holds for all $0 \le s < t$. Therefore, using (6.13)

$$2\int_0^t \left(kd_u - \frac{k^2 d_u^2}{c}\right)(C_L z + \mathbf{G}(0)v)^2 \le V(0) < \infty \quad \forall\, t \in \mathbb{R}_+\,. \tag{6.14}$$

Now recall that $c > k\lambda$ and $d_u(t) \in [0, \lambda]$ for a.e. $t \in \mathbb{R}_+$, so that

$$kd_u(t) - \frac{k^2 d_u(t)^2}{c} \ge kd_u(t)\left(1 - \frac{k\lambda}{c}\right) \ge k\frac{\delta}{\lambda}d_u(t)^2\,, \quad \text{a.e. } t \in \mathbb{R}_+\,, \tag{6.15}$$

where $\delta := 1 - k\lambda/c > 0$. Therefore, (6.14) gives

$$w \in L^2(\mathbb{R}_+, \mathbb{R})\,. \tag{6.16}$$

Using this in (6.5a) and appealing to the fact that $A^{-1}B$ is a bounded (and hence admissible) control operator for $\mathbf{T}$, we may use Lemma 3.1.4, part (2), to conclude that

$$\lim_{t\to\infty} \|z(t)\| = 0\,. \tag{6.17}$$

Since V is non-increasing (follows from (6.13) and (6.15)) and non-negative, we have that $V(t)$ converges as $t \to \infty$. Combining this with (6.17) and (6.12) implies that v^2 converges. By continuity of $\Phi(u)$, it follows that there exists a number $\Phi_\infty \in \mathbb{R}$ such that

$$\lim_{t\to\infty} (\Phi(u))(t) = \Phi_\infty\,.$$

We show that $\Phi_\infty = \Phi_r$. Setting

$$y_0(t) = (\Psi_\infty x_0)(t)\,, \quad y_1(t) = [\mathfrak{L}^{-1}(\mathbf{G}) \star (\Phi(u))](t)\,,$$

where $\star$ denotes convolution, we have

$$\dot{u}(t) = k[r - y_0(t) - y_1(t)]\,, \quad \text{a.e. } t \in \mathbb{R}\,. \tag{6.18}$$

Since $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_\infty$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$, it follows that

$$\lim_{t\to\infty} y_1(t) = \mathbf{G}(0)\Phi_\infty\,, \tag{6.19}$$

see [12], Theorem 6.1, part (ii), p. 96. Define a function $\tilde{y}_1 : \mathbb{R}_+ \to \mathbb{R}$ by setting

$$\tilde{y}_1(t) = r - y_1(t) = \mathbf{G}(0)\Phi_r - y_1(t)\,.$$

Seeking a contradiction, suppose that $\Phi_\infty \neq \Phi_r$. Then, either $\Phi_r > \Phi_\infty$ or $\Phi_r < \Phi_\infty$. If $\Phi_r > \Phi_\infty$, then by (6.19), there exists a number $\tau_0 \geq 0$ such that

$$\tilde{y}_1(t) \geq \frac{1}{2}\mathbf{G}(0)(\Phi_r - \Phi_\infty) > 0\,, \quad \forall\, t \geq \tau_0\,. \tag{6.20}$$

Integrating (6.18) yields

$$u(t) = u(\tau) + k\left(\int_\tau^t \tilde{y}_1(s)\,ds - \int_\tau^t y_0(s)\,ds\right)\,, \quad t \geq \tau \geq \tau_0\,. \tag{6.21}$$

By exponential stability, $y_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $y_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Therefore, for given $\varepsilon > 0$, there exists $\tau_\varepsilon \geq \tau_0$ such that

$$\int_{\tau_\varepsilon}^\infty |y_0(s)|\,ds \leq \frac{\varepsilon}{k}\,. \tag{6.22}$$

Defining $u_\varepsilon \in C(\mathbb{R}_+, \mathbb{R})$ by

$$u_\varepsilon(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq \tau_\varepsilon\,, \\ u(\tau_\varepsilon) + k\int_{\tau_\varepsilon}^t \tilde{y}_1(s)\,ds & \text{for } t > \tau_\varepsilon\,, \end{cases}$$

it follows from (6.20) that u_ε is ultimately non-decreasing, and moreover, by (6.21) and (6.22)

$$|u(t) - u_\varepsilon(t)| \leq \varepsilon\,, \quad \forall\, t \in \mathbb{R}_+\,,$$

showing that u is approximately ultimately non-decreasing. Since, by (6.20)–

(6.22), $u(t) \to \infty$ as $t \to \infty$, we may invoke (C5) to conclude that

$$\Phi_r > \Phi_\infty = \lim_{t\to\infty} (\Phi(u))(t) = \sup \mathrm{NVS}\,\Phi\,,$$

which is in contradiction to (6.4). If $\Phi_r < \Phi_\infty$, then a very similar argument shows that $-u$ is approximately ultimately non-decreasing and $\lim_{t\to\infty}(-u)(t) = \infty$. Invoking (C5) gives

$$\Phi_r < \Phi_\infty = \lim_{t\to\infty} (\Phi(u))(t) = \inf \mathrm{NVS}\,\Phi\,,$$

which again is in contradiction to (6.4). Therefore, we may conclude that $\Phi_\infty = \Phi_r$ and thus $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$, which is statement (1). Statement (2) follows from statement (1) and Lemma 3.1.4, part (1). For statement (3), we have

$$y(t) = C_L \mathbf{T}_t x_0 + (\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u))(t)\,. \tag{6.23}$$

By assumption $\mathfrak{L}^{-1}(\mathbf{G})$ is a finite signed Borel measure and since $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$ (by statement (1)), it follows from [12] (Theorem 6.1, part (ii), p. 96) that

$$\lim_{t\to\infty} [\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u)](t) = \mathbf{G}(0)\Phi_r = r\,.$$

Combining this with (6.23) shows that statement (3) holds. To prove statement (4), let $\Phi_r \in \operatorname{int} \mathrm{NVS}\,\Phi$. Then, boundedness of u follows immediately from statement (1) and (C6).

For statement (5), suppose that $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ is not a critical value of Φ. Since, by statement (4), u is bounded, there exists $d > 0$ and $T > 0$ such that

$$kd_u(t) \in [d, k\lambda] \quad \text{a.e. } t \geq T\,. \tag{6.24}$$

Choose $a \in (k\lambda, K)$ and let $\delta \in (0, d)$ be such that $a + \delta < K$. Define

$$\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} A^{-1}B \\ 1 \end{pmatrix}, \quad \tilde{C} := \begin{pmatrix} C, & \mathbf{G}(0) \end{pmatrix},$$

$$\kappa := \frac{a+\delta}{2} \in (k\lambda/2, K/2)\,, \quad \mathbf{G}_\kappa(s) := \frac{\mathbf{G}(s)}{s}\left(1 + \kappa\frac{\mathbf{G}(s)}{s}\right)^{-1}.$$

It is clear that $(\tilde{A}, \tilde{B}, \tilde{C}, 0)$ are the generating operators of a regular linear system with transfer function $\frac{\mathbf{G}(s)}{s}$. Let $\tilde{C}_L$ be the Lebesgue extension of $\tilde{C}$. Clearly, $\tilde{C}_L = (C_L, \mathbf{G}(0))$. Define $\tilde{A}_\kappa := \tilde{A} - \kappa\tilde{B}\tilde{C}_L$. Then, $\mathbf{G}_\kappa(s)$ is the transfer function of the closed-loop system $(\tilde{A}_\kappa, \tilde{B}, \tilde{C}, 0)$. It follows from [44] (see Theorem 7.2 in [44]) that $(\tilde{A}_\kappa, \tilde{B}, \tilde{C}, 0)$ are the generating operators of a regular linear system, where $\tilde{A}_\kappa$ has domain dom $(A) \times \mathbb{R}$, and hence $\tilde{A}_\kappa$ generates a strongly continuous semigroup $\mathbf{S}$.

Introduce $\tilde{z} = \begin{pmatrix} z \\ v \end{pmatrix}$; then, by (6.5), for a.e. $t \in \mathbb{R}_+$

$$\dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t) - kd_u(t)\tilde{B}\tilde{C}_L\tilde{z}(t) = \tilde{A}_\kappa\tilde{z}(t) - (kd_u(t) - \kappa)\tilde{B}\tilde{C}_L\tilde{z}(t)\,. \tag{6.25}$$

To establish that $\mathbf{S}$ is an exponentially stable semigroup, we consider, for $(z_0, v_0) \in X \times \mathbb{R}$, the system

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{v}_1(t) \end{pmatrix} = \tilde{A}_\kappa \begin{pmatrix} z_1(t) \\ v_1(t) \end{pmatrix}, \quad \begin{pmatrix} z_1(0) \\ v_1(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ v_0 \end{pmatrix}, \tag{6.26}$$

which is equivalent to the system

$$\dot{z}_1(t) \;=\; Az_1(t) + A^{-1}Bw_1(t)\,, \quad z_1(0) = z_0\,, \tag{6.27a}$$

$$\dot{v}_1(t) \;=\; w_1(t)\,, \quad v_1(0) = v_0\,, \tag{6.27b}$$

where

$$w_1(t) = -\kappa(C_L z_1(t) + \mathbf{G}(0)v_1(t))\,. \tag{6.28}$$

Taking the nonlinearity Φ to be the identity and $k = \kappa$ in (6.5), gives the same system as represented in (6.27) and thus by (6.16), $w_1 \in L^2(\mathbb{R}_+, \mathbb{R})$. An application of Lemma 3.1.4, part (2), to (6.27a), gives $z_1 \in L^2(\mathbb{R}_+, X)$ and an application of Lemma 3.1.4, part (3) to $(A, A^{-1}B, C)$ gives that $C_L z_1 \in L^2(\mathbb{R}_+, \mathbb{R})$. Finally, since $w_1, C_L z_1 \in L^2(\mathbb{R}_+, \mathbb{R})$ and $\mathbf{G}(0) \neq 0$, we may conclude from (6.28) that $v_1 \in L^2(\mathbb{R}_+, \mathbb{R})$ and so

$$\begin{pmatrix} z_1 \\ v_1 \end{pmatrix} \in L^2(\mathbb{R}_+, X \times \mathbb{R})\,.$$

Therefore, since

$$\mathbf{S}_t \begin{pmatrix} z_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} z_1(t) \\ v_1(t) \end{pmatrix},$$

by [8] (see Lemma 5.1.2 in [8]), $\mathbf{S}$ is exponentially stable.

We know from [28] that

$$\mathbf{G}_\kappa \in H^\infty(\mathbb{C}_0)\,. \tag{6.29}$$

Moreover, Lemma 3.10 in [28] yields

$$\|\mathbf{G}_\kappa\|_\infty := \sup_{s \in \mathbb{C}_0} |\mathbf{G}_\kappa(s)| = \frac{1}{\kappa}\,. \tag{6.30}$$

Setting

$$\gamma := \frac{a-\delta}{2}\,; \quad \Psi(f) := -(kd_u - \kappa)f\,, \quad \forall\, f \in F(\mathbb{R}_+, \mathbb{R})\,,$$

and using (6.24), we obtain $|kd_u(t) - \kappa| < \gamma$, for a.e. $t \geq T$ and therefore,

$$|(\Psi(f))(t)| \leq \gamma |f(t)|\,, \quad \forall\, f \in F(\mathbb{R}_+, \mathbb{R})\,, \quad \text{a.e. } t \geq T\,. \tag{6.31}$$

Clearly, $\kappa > \gamma > 0$, and hence by (6.30)

$$\gamma \|\mathbf{G}_\kappa\|_\infty < 1\,. \tag{6.32}$$

Let $\varepsilon > 0$ be sufficiently small such that the semigroups $e^{\varepsilon t}\mathbf{T}_t$ and $e^{\varepsilon t}\mathbf{S}_t$ are exponentially stable,

$$\mathbf{G}_\kappa \in H^\infty(\mathbb{C}_{-\varepsilon})\,, \tag{6.33}$$

and

$$\gamma \sup_{s \in \mathbb{C}_{-\varepsilon}} |\mathbf{G}_\kappa(s)| < 1\,. \tag{6.34}$$

For all sufficiently small $\varepsilon > 0$, (6.33) follows via a routine argument from (6.29) and the fact that $\mathbf{G} \in H^\infty(\mathbb{C}_{-\varepsilon})$, whilst (6.34) is a consequence of (6.32) and (6.33) combined with the fact that a holomorphic function which is bounded in an open vertical strip in the complex plane is uniformly continuous in any closed vertical substrip (see [7], p. 82).

From (6.25)

$$\dot{\tilde{z}}(t) = \tilde{A}_\kappa \tilde{z}(t) + \tilde{B}(\Psi(\tilde{C}_L \tilde{z}))(t)\,. \tag{6.35}$$

By (6.31), (6.24) and the fact that $w \in L^2(\mathbb{R}_+, \mathbb{R})$, we have $\Psi(\tilde{C}_L\tilde{z}) \in L^2(\mathbb{R}_+, \mathbb{R})$. Define the bounded operator H from $L^2(\mathbb{R}_+, \mathbb{R})$ to $L^2(\mathbb{R}_+, \mathbb{R})$ by setting

$$H(f) = \mathfrak{L}^{-1}(\mathbf{G}_\kappa \mathfrak{L}(f))\,, \quad \forall\, f \in L^2(\mathbb{R}_+, \mathbb{R})\,.$$

By (6.33), H restricts to a bounded operator from $L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$ to $L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$. The $L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$-induced operator norm of H is given by

$$\sup_{s \in \mathbb{C}_{-\varepsilon}} |\mathbf{G}_\kappa(s)| =: h\,. \tag{6.36}$$

Since $(\tilde{A}_\kappa, \tilde{B}, \tilde{C}, 0)$ is regular,

$$\tilde{C}_L \tilde{z}(t) = \tilde{C}_L \mathbf{S}_t \tilde{z}(0) + (H(\Psi(\tilde{C}_L \tilde{z})))(t)\,.$$

Taking the $L^2_{-\varepsilon}$-norm of $\mathbf{P}_t(\tilde{C}_L\tilde{z})$ (where $t \in \mathbb{R}_+$), using the causality of H and estimating gives

$$\left(\int_0^t |e^{\varepsilon\tau}\tilde{C}_L\tilde{z}(\tau)|^2\,d\tau\right)^{1/2} \leq \left(\int_0^\infty |e^{\varepsilon\tau}\tilde{C}_L\mathbf{S}_t\tilde{z}(0)|^2\,d\tau\right)^{1/2} + h\left(\int_0^t |e^{\varepsilon\tau}(\Psi(\tilde{C}_L\tilde{z}))(\tau)|^2\,d\tau\right)^{1/2}, \ \forall t \geq 0\,. \quad (6.37)$$

Combining (6.31), (6.37) and using (3.6) applied to the exponentially stable regular system $(\tilde{A}_\kappa + \varepsilon I, \tilde{B}, \tilde{C}, 0)$, we may conclude that there exists $N_1 > 0$ such that

$$\left(\int_0^t |e^{\varepsilon\tau}\tilde{C}_L\tilde{z}(\tau)|^2\,d\tau\right)^{1/2} \leq N_1 + \gamma h\left(\int_0^t |e^{\varepsilon\tau}\tilde{C}_L\tilde{z}(\tau)|^2\,d\tau\right)^{1/2}, \quad \forall t \geq 0\,.$$

By (6.34) and (6.36), $\gamma h < 1$, and therefore, $\tilde{C}_L\tilde{z} \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$. Thus

$$w \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})\,. \quad (6.38)$$

Define $z_\varepsilon(t) = \exp(\varepsilon t)z(t)$ and $w_\varepsilon(t) = \exp(\varepsilon t)w(t)$. Then using (6.5a)

$$\dot{z}_\varepsilon(t) = (A + \varepsilon I)z_\varepsilon(t) + A^{-1}Bw_\varepsilon(t)\,,$$

for a.e. $t \in \mathbb{R}_+$ and therefore since $e^{\varepsilon t}\mathbf{T}_t$ is exponentially stable, by Lemma 3.1.4, part (2), z_ε is bounded. Let $\rho \in (0, \varepsilon)$; we show that $\exp(\rho t)v(t)$ is bounded. By (6.38), an application of Lemma 3.1.5 to $(A, A^{-1}B, C, 0)$ gives, $C_L z \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$, and therefore since $C_L z + \mathbf{G}(0)v = \tilde{C}_L\tilde{z} \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$,

$$v \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})\,. \quad (6.39)$$

Define $v_\rho(t) = e^{\rho t}v(t)$ and $w_\rho(t) = e^{\rho t}w(t)$, then using (6.5b), for a.e. $t \in \mathbb{R}_+$

$$\dot{v}_\rho(t) = \rho v_\rho(t) + w_\rho(t)\,,$$

and therefore by (6.38) and (6.39), $\dot{v}_\rho \in L^2_{\rho-\varepsilon}(\mathbb{R}_+, \mathbb{R}) \subset L^1(\mathbb{R}_+, \mathbb{R})$ and thus v_ρ is bounded. By the boundedness of v_ρ, the convergence in statement (1) is of order $\exp(-\rho t)$. Define $x_\rho(t) = e^{\rho t}(x(t) + A^{-1}B\Phi_r)$, then for a.e. $t \in \mathbb{R}_+$, $\dot{x}_\rho(t) = (A + \rho I)x_\rho(t) + Bv_\rho(t)$. Therefore, by Lemma 3.1.4, part (3), since v_ρ is a bounded input and $e^{\rho t}\mathbf{T}_t$ is an exponentially stable semigroup, x_ρ is bounded. Thus the convergence in statement (2) is of order $\exp(\rho t)$.

Suppose $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}^\alpha_{\mathrm{f}}(\mathbb{R}_+)$ for some $\alpha < 0$ and let $\rho \in (0, -\alpha)$ be such that the convergence in statements (1) and (2) is of order $\exp(-\rho t)$. We show that the convergence in statement (3) is also of order $\exp(-\rho t)$. Recall that U

denotes the unit-step function. We have for all $t \in \mathbb{R}_+$

$$|r - y(t) + (\Psi_\infty x_0)(t)| \leq |[\mu \star (\Phi(u) - \Phi_r U)](t)| + |\Phi_r[(\mu \star U)(t) - \mathbf{G}(0)]| \,. \quad (6.40)$$

We see that

$$e^{\rho t}[\mu \star (\Phi(u) - \Phi_r U)](t) = \int_0^t ((\Phi(u))(t-s) - \Phi_r)e^{\rho(t-s)}e^{\rho s}\, d\mu(s)\,,$$

and since $t \mapsto e^{\rho t}|(\Phi(u))(t) - \Phi_r|$ is a bounded function and $E \mapsto \int_E e^{\rho t}\, d\mu(t)$ belongs to $\mathcal{M}_\mathrm{f}(\mathbb{R}_+)$, we may conclude that the function $t \mapsto e^{\rho t}[\mu \star (\Phi(u) - \Phi_r U)](t)$ is bounded on $\mathbb{R}_+$.

Since $\mu \in \mathcal{M}_\mathrm{f}^\alpha(\mathbb{R}_+)$, the total variation $|\mu|$ of μ also belongs to $\mathcal{M}_\mathrm{f}^\alpha(\mathbb{R}_+)$. Hence

$$\begin{aligned} |e^{\rho t}\Phi_r[(\mu \star U)(t) - \mathbf{G}(0)]| &= \left| e^{\rho t}\Phi_r\left(\int_0^t d\mu(s) - \int_0^\infty d\mu(s)\right)\right| \\ &\leq |\Phi_r| \int_t^\infty e^{\rho t}\, d|\mu|(s) < \infty\,, \end{aligned}$$

showing that the function $t \mapsto e^{\rho t}[(\mu \star U)(t) - \mathbf{G}(0)]$ is bounded on $\mathbb{R}_+$. Consequently, appealing to (6.40), we deduce that the function

$$\mathbb{R}_+ \to \mathbb{R}, \quad t \mapsto e^{\rho t}|r - y(t) + (\Psi_\infty x_0)(t)|$$

is bounded.

Finally for statement (6), we recall that $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$ (by statement (1)) and that $\Phi(u) - \Phi_r \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$ (by (6.39)). Applying Lemma 3.1.5, we may conclude that $y - \mathbf{G}(0)\Phi_r = y - r \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R})$. Thus $\dot{u} \in L^2_{-\varepsilon}(\mathbb{R}_+, \mathbb{R}) \subset L^1(\mathbb{R}_+, \mathbb{R})$ and so u converges to a finite limit. $\square$

Remark 6.1.4 We see from the proof of Theorem 6.1.2 that (C6) is only needed for statement (4) and that nowhere do we require the operator Φ to be rate independent. All we need is that Φ is causal and satisfies the property expressed in statement (1) of Theorem 4.1.2 (this property is a consequence of rate independence). $\diamond$

One of the conditions imposed in Theorem 6.1.2 is that $r/\mathbf{G}(0) \in \mathrm{clos}\,(\mathrm{NVS}\,\Phi)$. The following proposition shows that this condition is close to being necessary for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\Phi(u)$, then $r/\mathbf{G}(0) \in \mathrm{clos}\,(\mathrm{NVS}\,\Phi)$.

Proposition 6.1.5 *Let $\lambda > 0$ and $r \in \mathbb{R}$. Suppose that $(A, B, C, D) \in \mathcal{L}$ and $\Phi \in \mathcal{N}_c(\lambda)$. If there exist an initial condition $x_0 \in X$ and a continuous function*

$u : \mathbb{R}_+ \to \mathbb{R}$ such that $\Phi(u)$ is bounded and

$$\lim_{t\to\infty} [C_L x(t) + D(\Phi(u))(t)] = r \,,$$

where $x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B(\Phi(u))(\tau)\, d\tau$, then $r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$.

Proposition 6.1.5 can be proved in a similar way to Proposition 3.4 in [10].

6.2 Example: controlled diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $[0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_1 \in (0, 1)$, via an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, as defined below) and delayed (delay $T \geq 0$) pointwise scalar observation (output at point $x_2 \in (x_1, 1)$).

We formally write this single-input, single-output system as

$$\begin{aligned} z_t(t,x) &= \kappa z_{xx}(t,x) + \delta(x - x_1)(\Phi(u))(t) \,, \\ y(t) &= z(t - T, x_2) \,, \end{aligned}$$

with boundary conditions

$$z(t, 0) = 0 = z(t, 1) \,, \quad \forall\, t > 0 \,.$$

For simplicity, we assume zero initial conditions

$$z(t, x) = 0 \,, \quad \forall\, (t, x) \in [-T, 0] \times [0, 1] \,.$$

These equations model the problem of heating a rod of unit length whose ends are kept at zero temperature and which is initially zero temperature across its length. We want to raise the temperature at a point x_2 along its length, to value r, by applying heat at a point x_1 along its length. The function $z(t, \cdot)$ is the temperature profile along the length of the rod at time $t \in \mathbb{R}_+$.

With input $(\Phi(u))(\cdot)$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sT} \sinh\left(x_1\sqrt{s/\kappa}\right) \sinh\left((1 - x_2)\sqrt{s/\kappa}\right)}{\kappa\sqrt{s/\kappa}\sinh\sqrt{s/\kappa}} \,.$$

It is not difficult to show that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for any $\alpha > -\kappa\pi^2$ (see Appendix 6 for details). A detailed analysis (see [25]) shows that K, defined by (6.3), satisfies

$$K = \frac{1}{|\mathbf{G}'(0)|} = \frac{6\kappa^2}{x_1(1-x_2)(6T\kappa + 1 - x_1^2 - (1-x_2)^2)}.$$

Therefore, by Theorem 6.1.2, if $\Phi \in \mathcal{N}_c(\lambda)$ for some $\lambda > 0$ and $k \in (0, K/\lambda)$, the integral control, $\dot{u}(t) = k[r - y(t)]$, with $u(0) = u_0$, guarantees asymptotic tracking of all feasible constant reference signals r. For purposes of illustration, we adopt the following values

$$\kappa = 0.1, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad T = 1,$$

and so $K = 243/620 \approx 0.3919$.

We consider relay, Prandtl and backlash hysteresis operators:

(a) Let $\Phi = \mathcal{R}_\xi$ be a relay hysteresis operator as defined in (4.12), where $\xi = 0$, $a_1 = -1$, $a_2 = 1$, $\rho_1(u) = \sqrt{u + 1.1}$ and $\rho_2(u) = \sqrt{0.1} + \sqrt{2.1} - \sqrt{1.1 - u}$. Then $\Phi \in \mathcal{N}_c(\lambda)$ where $\lambda = 1.6$, $\mathrm{NVS}\,\Phi = \mathrm{im}\,\rho_1 \cup \mathrm{im}\,\rho_2 = \mathbb{R}$ and $K/\lambda \approx 0.245$. We take $r = 1.42$ and $u_0 = 0$. Then

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 1.278 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi).$$

In each of the following three cases of admissible controller gains

(i) $k = 0.244$ (solid), (ii) $k = 0.17$ (dashdot), (iii) $k = 0.1$ (dotted),

Figure 16 depicts the output behaviour of the system under integral control, Figure 17 depicts the corresponding control input and Figure 18 shows the input of the relay hysteresis operator. Figure 19 illustrates the evolution of the temperature profile $z(t, \cdot)$ in case (i). Since Φ_r is not a critical value of $\mathcal{R}_\xi$, statements (5) and (6) of Theorem 6.1.2 hold and therefore the convergence seen in Figures 16, 17 and 19 is of exponential order and u converges in Figure 18. In particular for (i), $\lim_{t\to\infty} u(t) = \rho_1^{-1}(\Phi_r)$ and for (ii) and (iii), $\lim_{t\to\infty} u(t) = \rho_2^{-1}(\Phi_r)$.

(b) Let $\Phi = \mathcal{P}_\zeta$ be a Prandtl operator, as defined in (4.27), where $p = (1/10)\chi_{[2,5]}$ and $\zeta \equiv 0$. Then $\Phi \in \mathcal{N}_c(\lambda)$ where $\lambda = 1$, $\mathrm{NVS}\,\Phi = [-10.5, 10.5]$ and $K/\lambda = K \approx 0.3919$. We take $r = 1$ and $u_0 = 2$. Then

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 0.9 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi).$$

In each of the following three cases of admissible controller gains

$$\text{(i) } k = 0.39 \text{ (solid)}, \quad \text{(ii) } k = 0.25 \text{ (dashdot)}, \quad \text{(iii) } k = 0.1 \text{ (dotted)},$$

Figure 20 depicts the output behaviour of the system under integral control, Figure 21 depicts the corresponding control input and Figure 22 shows the input of the Prandtl operator. We know from Proposition 5.2.19 that Φ_r is not a critical numerical value of $\mathcal{P}_\zeta$ and therefore u, the input to the hysteresis operator, converges, and y and $\Phi(u)$ (as well as the state) converge with exponential order. We see from Figure 22 that in each of the three cases, u converges to a different value because of the formation of "different" hysteresis loops.

(c) Let $\Phi = \mathcal{B}_{0.5,0}$ be a standard backlash hysteresis operator as defined in Section 4.3. Then $\Phi \in \mathcal{N}_c(\lambda)$ where $\lambda = 1$, $\mathrm{NVS}\,\Phi = \mathbb{R}$ and $K/\lambda = K \approx 0.3919$. We take $r = 1$ and $u_0 = 0$. Then

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 0.9 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)\,.$$

In each of the following three cases of admissible controller gain

$$\text{(i) } k = 0.39 \text{ (solid)}, \quad \text{(ii) } k = 0.25 \text{ (dashdot)}, \quad \text{(iii) } k = 0.1 \text{ (dotted)},$$

Figure 23 depicts the output behaviour of the system under integral control, Figure 24 depicts the corresponding control input and Figure 25 shows the input of the backlash operator. We remark that the convergence of $u(t)$ as $t \to \infty$ is not guaranteed by Theorem 6.1.2 and in fact it seems that u does not converge in two of the cases.

Figures 16–25 were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 20, was adopted to model the diffusion process.

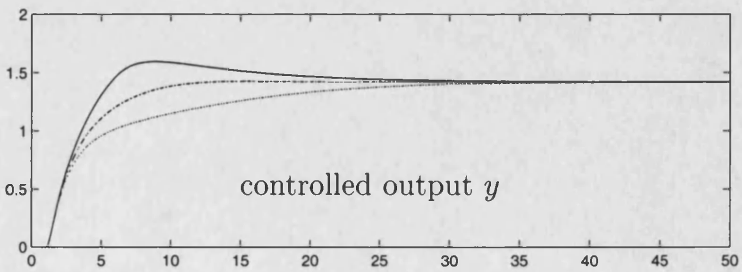

Figure 16: Controlled output

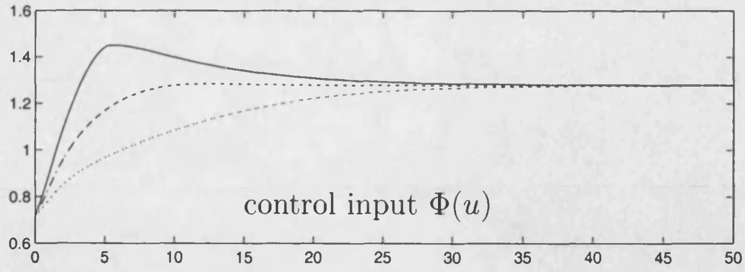

Figure 17: Control input

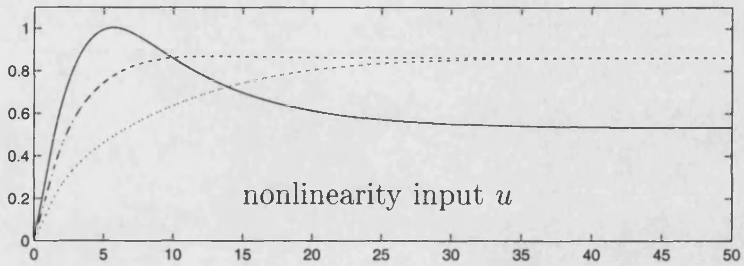

Figure 18: Input of relay operator

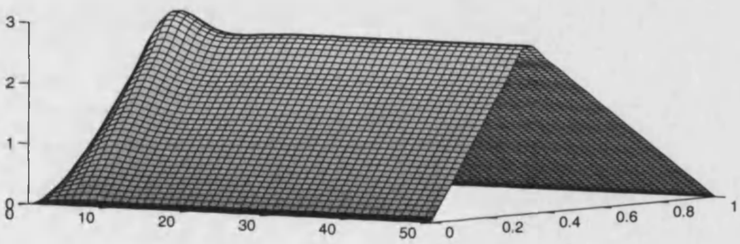

Figure 19: Temperature profile in case (i) (k=0.244)

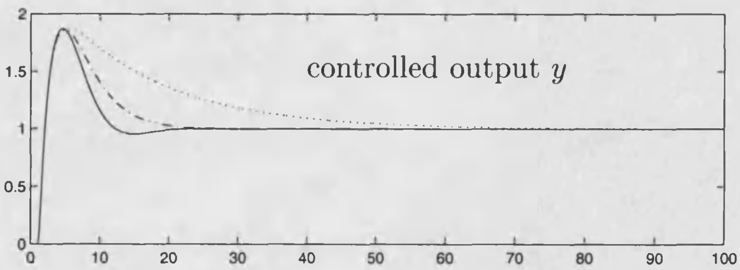

Figure 20: Controlled output

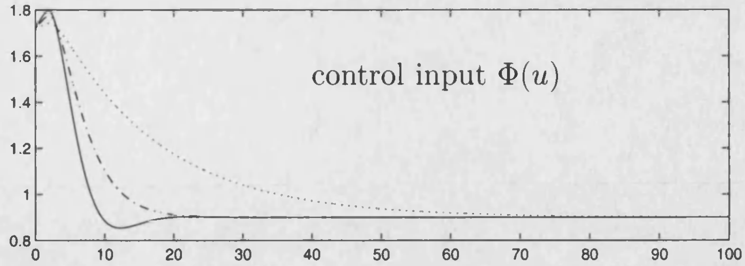

Figure 21: Control input

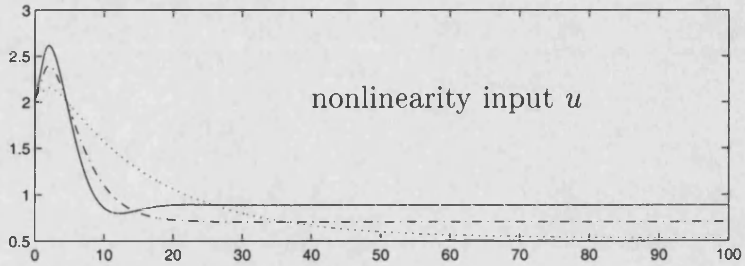

Figure 22: Input of Prandtl operator

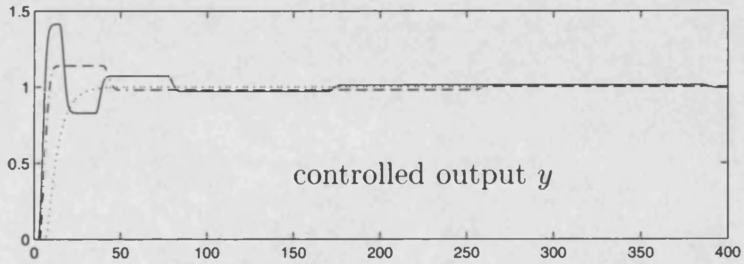

Figure 23: Controlled output

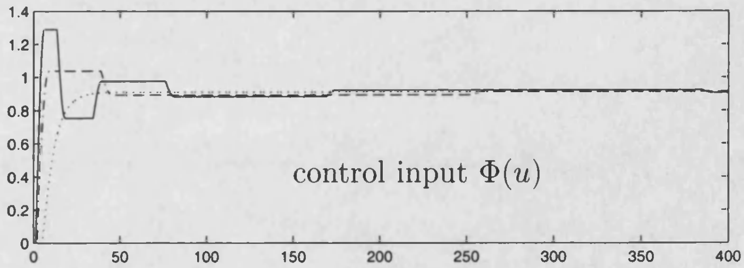

Figure 24: Control input

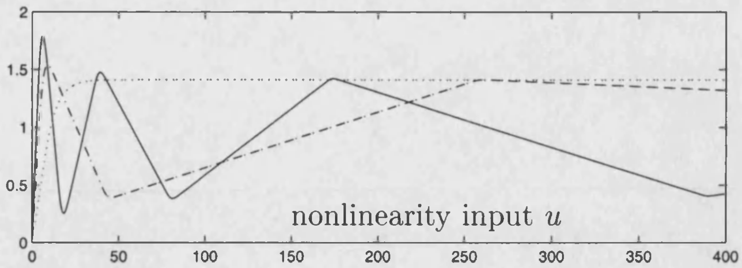

Figure 25: Input of backlash operator

6.3 Example: controlled damped wave equation with output delay

Consider a damped wave equation (with Dirichlet boundary conditions), on the one-dimensional spatial domain $[0,1]$, with scalar nonlinear pointwise control action (applied at point $x_1 \in (0,1)$, via an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, as defined below) and delayed (delay $T \geq 0$) scalar observation generated by a spacial averaging of the delayed state over an ε-neighbourhood of a point $x_2 \in (0, x_1)$, where $\varepsilon \in (0, \min(x_2, x_1 - x_2))$.

We formally write this single-input, single-output system as

$$\begin{aligned} z_{tt}(t,x) &= \kappa z_{xx}(t,x) - b z_t(t,x) + \delta(x - x_1)(\Phi(u))(t)\,, \\ y(t) &= \frac{1}{2\varepsilon}\int_{x_2-\varepsilon}^{x_2+\varepsilon} z(t-T,x)\,dx\,, \end{aligned}$$

with boundary conditions

$$z(t,0) = 0 = z(t,1)\,, \quad \forall\, t > 0\,.$$

We take zero initial conditions:

$$z(t,x) = 0 = z_t(t,x)\,, \quad \forall\,(t,x) \in [-T,0] \times [0,1]\,.$$

With input $(\Phi(u))(\cdot)$ and output $y(\cdot)$, this example qualifies as a regular linear system with bounded observation operator. To find the transfer function we take Laplace transforms of the partial differential equation (with input $v = \Phi(u)$). Writing $\hat{f}$ for the Laplace transform of f, this yields a boundary value problem for $\hat{z}(s,x)$, where we regard s as a parameter:

$$\begin{aligned}
&s^2\hat{z}(s,x) = \kappa\hat{z}_{xx}(s,x) - bs\hat{z}(s,x) + \delta(x-x_1)\hat{v}(s)\,,\\
&\hat{z}(s,0) = 0 = \hat{z}(s,1)\,.
\end{aligned}$$

This can be rewritten as a first-order system

$$\frac{d}{dx}\begin{pmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{pmatrix}(s,x) = \begin{pmatrix} 0 & 1 \\ \nu(s) & 0 \end{pmatrix}\begin{pmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{pmatrix}(s,x) - \begin{pmatrix} 0 \\ \frac{1}{\kappa} \end{pmatrix}\delta(x-x_1)\hat{v}(s)\,,$$

where $\nu(s) = \frac{s^2+bs}{\kappa}$. For $s \neq 0$ the above first-order system has solution

$$\begin{aligned}
\begin{pmatrix} \hat{z} \\ \frac{d\hat{z}}{dx} \end{pmatrix}(s,x) &= \begin{pmatrix} \cosh(\sqrt{\nu(s)}x) & \frac{1}{\sqrt{\nu(s)}}\sinh(\sqrt{\nu(s)}x) \\ \sqrt{\nu(s)}\sinh(\sqrt{\nu(s)}x) & \cosh(\sqrt{\nu(s)}x) \end{pmatrix}\begin{pmatrix} 0 \\ \frac{d\hat{z}}{dx}(s,0) \end{pmatrix} \\
&\quad -\frac{1}{\kappa}\int_0^x \begin{pmatrix} \frac{1}{\sqrt{\nu(s)}}\sinh(\sqrt{\nu(s)}(x-\gamma)) \\ \cosh(\sqrt{\nu(s)}(x-\gamma)) \end{pmatrix}\delta(\gamma-x_1)\hat{v}(s)\,d\gamma\,.
\end{aligned}$$

In addition we have

$$0 = \hat{z}(s,0) = \frac{1}{\sqrt{\nu(s)}}\sinh(\sqrt{\nu(s)})\frac{d\hat{z}}{dx}(s,0) - \frac{1}{\kappa\sqrt{\nu(s)}}\sinh(\sqrt{\nu(s)}(1-x_1))\hat{v}(s)\,.$$

Thus

$$\frac{d\hat{z}}{dx}(s,0) = \frac{\sinh(\sqrt{\nu(s)}(1-x_1))}{\kappa\sinh(\sqrt{\nu(s)})}\hat{v}(s)\,,$$

and therefore

$$\hat{z}(s,x) = \frac{\sinh(x\sqrt{\nu(s)})\sinh((1-x_1)\sqrt{\nu(s)})}{\kappa\sqrt{\nu(s)}\sinh(\sqrt{\nu(s)})}\,.$$

for all $x \in (0,x_1)$. Thus,

$$\begin{aligned}
&\int_{x_2-\varepsilon}^{x_2+\varepsilon}\hat{z}(s,x)\,dx \\
&\quad = \frac{2\varepsilon\sinh((1-x_1)\sqrt{\nu(s)})\hat{v}(s)}{\kappa\sqrt{\nu(s)}\sinh(\sqrt{\nu(s)})}\int_{x_2-\varepsilon}^{x_2+\varepsilon}\sinh(x\sqrt{\nu(s)})\,dx \\
&\quad = \frac{[\cosh((x_2-\varepsilon)\sqrt{\nu(s)}) - \cosh((x_2+\varepsilon)\sqrt{\nu(s)})]\sinh((1-x_1)\sqrt{\nu(s)})\hat{v}(s)}{\kappa\nu(s)\sinh(\sqrt{\nu(s)})}\,.
\end{aligned}$$

Hence, applying Fubini's theorem,

$$\mathbf{G}(s) = \frac{e^{-sT}[\cosh((x_2-\varepsilon)\sqrt{\nu(s)}) - \cosh((x_2+\varepsilon)\sqrt{\nu(s)})]\sinh((1-x_1)\sqrt{\nu(s)})}{2\varepsilon\kappa\nu(s)\sinh(\sqrt{\nu(s)})}\,.$$

Since we have a bounded observation operator, $\mathfrak{L}^{-1}(G) \in \mathcal{M}_f(\mathbb{R}_+)$ (see Lemma 2.3 in [24]).

From [25] we know that $K \geq 1/\|\mathbf{E}\|_\infty$ where

$$\mathbf{E}(s) := \frac{1}{s}(\mathbf{G}(s) - \mathbf{G}(0))\,.$$

For purposes of illustration, we adopt the following values

$$\kappa = 1, \quad b = 0.1, \quad x_1 = \frac{2}{3}, \quad x_2 = \frac{1}{3}, \quad T = 1, \quad \varepsilon = 0.01\,,$$

and so $K \geq 1/\|\mathbf{E}\|_\infty \approx 0.657$.

Therefore if $\Phi \in \mathcal{N}_c(\lambda)$, for some $\lambda > 0$, by Theorem 6.1.2, for each $k \in (0, K/\lambda)$, the integral control $\dot{u}(t) = k[r - y(t)]$, with $u(0) = 5$, guarantees asymptotic tracking of all feasible constant reference signals r.

Let $\Phi = \mathcal{B}_{0.5,0}$ be a standard backlash hysteresis operator as defined in Section 4.3. Then $\Phi \in \mathcal{C}(1)$ and $\mathrm{NVS}\,\Phi = \mathbb{R}$. For reference value $r = 1$

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1 - x_2)} = 0.9 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)\,.$$

In each of the following three cases of admissible controller gain

(i) $k = 0.65$ (solid), (ii) $k = 0.5$ (dotdash), (iii) $k = 0.35$ (dotted),

Figure 26 depicts the output behaviour of the system under integral control, Figure 27 depicts the corresponding control input and Figure 28 shows the input of the backlash operator. We remark that although Theorem 6.1.2 does not guarantee the convergence of u, it appears that in this example u does converge.

Figures 26–28 were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 20, was adopted to model the diffusion process.

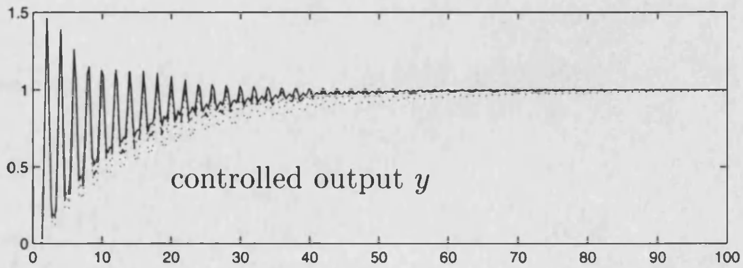

Figure 26: Controlled output

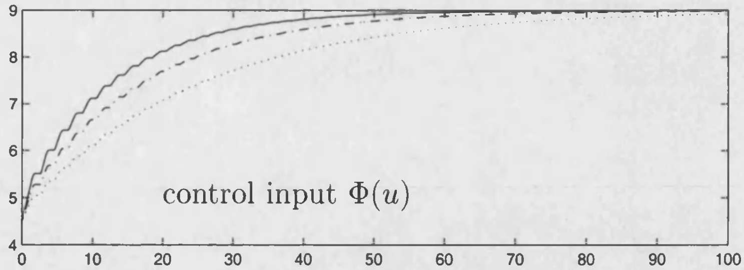

Figure 27: Control input

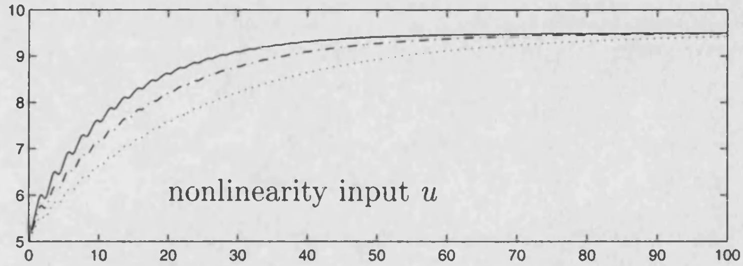

Figure 28: Input of backlash operator

6.4 Notes and references

As previously remarked, Theorem 6.1.2 is an extension of the main result in [26]. However, the method used here to prove statements (1)–(4) of Theorem 6.1.2 is different from that used in [26]. In [26], complex stability radius results were used to show that the generating operators of the regular system satisfy a certain Riccati equation. Here we made use of the positive-real Riccati equation theory for Pritchard-Salamon systems developed by van Keulen [15]. This new method (suggested by H. Logemann) seems to be more natural than the approach in [26]. Additionally, we remark that the results contained in statements (5) and (6) of Theorem 6.1.2 are not an extension of any results contained in [26] and are in fact new even for static nonlinearities. Theorem 6.1.2 is also new in the case when (A, B, C, D) is a finite-dimensional system. The results contained in statements (1)–(4) of Theorem 6.1.2 first appeared in [19] by Logemann and Mawby.

Chapter 7

Low-gain control of discrete-time linear systems subject to input hysteresis

7.1 Discrete-time integral control in the presence of input nonlinearities in $\mathcal{N}_d(\lambda)$

Consider a single-input, single-output, discrete-time system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n)\,, \quad x(0) = x_0 \in X\,, && (7.1a) \\ y(n) &= Cx(n) + Du(n)\,, && (7.1b) \end{aligned}$$

evolving on a real Hilbert space X (with norm $\|\cdot\|$). Here $A \in L(X)$, $B \in L(\mathbb{R}, X)$, $C \in L(X, \mathbb{R})$ and $D \in \mathbb{R}$. A system of the form (7.1) is called *power-stable* if A is power-stable, i.e. there exist $M > 0$ and $\theta \in (0,1)$ such that

$$\|A^n\| \le M\theta^n\,, \quad \forall\, n \in \mathbb{Z}_+\,,$$

where $\|\cdot\|$ denotes the operator norm on $L(X)$ induced by the norm $\|\cdot\|$ on X. The transfer function $\mathbf{G}$ of (7.1) is given by

$$\mathbf{G}(z) = C(zI - A)^{-1}B + D\,.$$

For future reference we state the following simple lemma. It is the discrete-time analogy of Lemma 3.1.4.

Lemma 7.1.1 *Assume that A is power-stable. Then the following statements hold.*

(1) If $u \in l^\infty(\mathbb{Z}_+, \mathbb{R})$ is such that $\lim_{n\to\infty} u(n) = u_\infty$ exists, then, for all $x_0 \in X$, the state $x(\cdot)$ given by (7.1a) satisfies

$$\lim_{n\to\infty} x(n) = (I - A)^{-1} B u_\infty \,.$$

(2) If $u \in l^2(\mathbb{Z}_+, \mathbb{R})$, then, for all $x_0 \in X$, the state $x(\cdot)$ given by (7.1a) satisfies

$$\lim_{n\to\infty} x(n) = 0 \,, \quad x \in l^2(\mathbb{Z}_+, \mathbb{R}) \,.$$

Suppose that system (7.1) is subject to a causal input nonlinearity $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$, yielding the nonlinear system

$$\begin{aligned} x(n+1) &= Ax(n) + B(\Phi(u))(n) \,, \quad x(0) = x_0 \in X \,, && (7.2a) \\ y(n) &= Cx(n) + D(\Phi(u))(n) \,. && (7.2b) \end{aligned}$$

Denoting the reference value by r, the control law

$$u(n+1) = u(n) + k(r - y(n)) \,,$$

where k is a real parameter, then leads to the following nonlinear system of difference equations

$$\begin{aligned} x(n+1) &= Ax(n) + B(\Phi(u))(n) \,, \quad x(0) = x_0 \in X \,, && (7.3a) \\ u(n+1) &= u(n) + k(r - Cx(n) - D(\Phi(u))(n)) \,, \quad u(0) = u_0 \in \mathbb{R} \,. && (7.3b) \end{aligned}$$

If $\mathbf{G} \in H^\infty(\mathbb{E}_\alpha)$ for some $\alpha \in (0,1)$ (which is the case if (7.1) is power-stable) and $\mathbf{G}(1) > 0$, then it can be shown that

$$1 + k \operatorname{Re} \frac{\mathbf{G}(z)}{z-1} \geq 0 \,, \quad \forall\, z \in \mathbb{E}_1 \,, \tag{7.4}$$

for all sufficiently small $k > 0$, see Lemma 2.9 in [27]. We define

$$K := \sup\{k > 0 \mid (7.4) \text{ holds}\} \,. \tag{7.5}$$

Before we formulate the main result of the section, we derive a lower bound for K. To this end it will be convenient to introduce the following auxiliary transfer function

$$\mathbf{E}(z) := \frac{\mathbf{G}(z) - \mathbf{G}(1)}{z-1} \,.$$

The above definition makes sense for all $z \neq 1$ for which $\mathbf{G}(z)$ is defined. If $\mathbf{G}(z)$ is holomorphic at 1, then we set $\mathbf{E}(1) = \mathbf{G}'(1)$.

Lemma 7.1.2 *Assume that* $\mathbf{G} \in H^\infty(\mathbb{E}_\alpha)$ *for some* $\alpha \in (0,1)$ *and that* $\mathbf{G}(1) > 0$ *and let* $k > 0$. *Then the following statements are equivalent*

(1) $1 + k \operatorname{Re} \frac{\mathbf{G}(z)}{z-1} \geq 0$, for all $z \in \mathbb{E}_1$,

(2) $1 + k \operatorname{Re} \frac{\mathbf{G}(e^{i\theta})}{e^{i\theta}-1} \geq 0$, for all $\theta \in (0, 2\pi)$,

(3) $1 + k\,(\operatorname{Re} \mathbf{E}(e^{i\theta}) - \mathbf{G}(1)/2) \geq 0$, for all $\theta \in [0, 2\pi)$,

(4) $1 + k\,(\operatorname{Re} \mathbf{E}(z) - \mathbf{G}(1)/2) \geq 0$, for all $z \in \mathbb{E}_1$.

Proof: Trivially, (1) implies (2), and since $\mathbf{G}(1)$ is real and

$$\operatorname{Re} \frac{1 + e^{i\theta}}{1 - e^{i\theta}} = 0\,, \quad \forall\, \theta \in (0, 2\pi)\,,$$

(2) implies (3). In order to show that (4) follows from (3), assume that (3) holds. Define $f(z) = \exp(-(1 + k\mathbf{E}(z) - k\mathbf{G}(1)/2))$ for all $z \in \mathbb{E}_\alpha$. Now f is holomorphic on $\mathbb{E}_1$ since $\mathbf{G}$ is holomorphic on $\mathbb{E}_1$ and f is bounded and continuous on $\operatorname{clos} \mathbb{E}_1$ since $\mathbf{G} \in H^\infty(\mathbb{E}_\alpha)$. Therefore by the maximum modulus theorem

$$\|f\|_\infty = \sup_{z \in \mathbb{E}_1} |f(z)| = \sup_{\theta \in [0, 2\pi)} |f(e^{i\theta})|\,,$$

which implies that (using (3))

$$-\inf_{z \in \mathbb{E}_1} (1 + k \operatorname{Re} \mathbf{E}(z) - k\mathbf{G}(1)/2) = -\inf_{\theta \in [0, 2\pi)} (1 + k \operatorname{Re} \mathbf{E}(e^{i\theta}) - k\mathbf{G}(1)/2) \leq 0\,.$$

Thus (4) holds. Finally, since $\mathbf{G}(1) > 0$, we have that $\mathbf{G}(1) \operatorname{Re} \frac{z+1}{z-1} > 0$ for all $z \in \mathbb{E}_1$, and therefore (1) is implied by (4). □

The following corollary provides a lower bound and an upper bound for K in terms of the transfer function $\mathbf{E}$.

Corollary 7.1.3 *Assume that* $\mathbf{G} \in H^\infty(\mathbb{E}_\alpha)$ *for some* $\alpha \in (0,1)$ *and that* $\mathbf{G}(1) > 0$. *Then*

$$\frac{1}{\|\mathbf{E}\|_\infty + \mathbf{G}(1)/2} \leq K \leq \begin{cases} \frac{1}{|\operatorname{Re} \mathbf{E}(1)| + \mathbf{G}(1)/2} & \text{if } \operatorname{Re} \mathbf{E}(1) \leq 0 \\ \infty & \text{if } \operatorname{Re} \mathbf{E}(1) > 0\,. \end{cases} \tag{7.6}$$

Proof: For $k > 0$ we have that

$$1 + k \operatorname{Re} \mathbf{E}(z) - k\mathbf{G}(1)/2 \geq 1 - k(\|\mathbf{E}\|_\infty + \mathbf{G}(1)/2)\,, \quad \forall\, z \in \mathbb{E}_1\,.$$

Combining this with Lemma 7.1.2, we see that

$$K \geq \frac{1}{\|\mathbf{E}\|_\infty + \mathbf{G}(1)/2},$$

which is the first inequality in (7.6). Moreover, using Lemma 7.1.2 again, it follows from the definition of K that $1 + K(\operatorname{Re}\mathbf{E}(1) - \mathbf{G}(1)/2) \geq 0$. If $\operatorname{Re}\mathbf{E}(1) \leq 0$, we may conclude that

$$K \leq \frac{1}{|\operatorname{Re}\mathbf{E}(1)| + \mathbf{G}(1)/2},$$

yielding the second inequality in (7.6). □

We now state the main result of this section.

Theorem 7.1.4 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_d(\lambda)$, A is power-stable, $\mathbf{G}(1) > 0$, $k \in (0, K/\lambda)$ and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(1) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$. Then for all $(x_0, u_0) \in X \times \mathbb{R}$ the solution (x, u) of (7.3) satisfies*

(1) $\lim_{n\to\infty}(\Phi(u))(n) = \Phi_r$,

(2) $\lim_{n\to\infty} x(n) = (I - A)^{-1}B\Phi_r$,

(3) $\lim_{n\to\infty} y(n) = r$, *where* $y(n) = Cx(n) + D(\Phi(u))(n)$,

(4) if $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$, *then u is bounded,*

(5) if $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$ *and Φ_r is not a critical numerical value of Φ, then the convergence in (1) and (2) (and hence in (3)) is of order ρ^{-n} for some $\rho > 1$ (in the sense that the functions $\rho^n((\Phi(u))(n) - \Phi_r)$, $\rho^n(x(n) - (I - A)^{-1}B\Phi_r)$ and $\rho^n(y(n) - r)$ are bounded),*

(6) if $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$ *and Φ_r is not a critical numerical value of Φ, then there exists $u_\infty \in \mathbb{R}$ such that* $\lim_{n\to\infty} u(n) = u_\infty$.

Proof: Denote the solution of (7.3) by (x, u) and introduce new variables by defining

$$z(n) := x(n) - (I - A)^{-1}B(\Phi(u))(n), \quad v(n) := (\Phi(u))(n) - \Phi_r; \quad \forall n \in \mathbb{Z}_+.$$

For convenience we define $d_u = \delta\Phi(u)$ (recall $\delta\Phi$ from Definition 5.3.6). Then $(\Phi(u))(n+1) - (\Phi(u))(n) = d_u(n)(u(n+1) - u(n))$ for all $n \in \mathbb{Z}_+$. Using the identity $A(I - A)^{-1} = (I - A)^{-1} - I$, a straightforward calculation yields

$$\begin{aligned} z(n+1) &= Az(n) - (I - A)^{-1}Bw(n), \quad z(0) = z_0 && (7.7a)\\ v(n+1) &= v(n) + w(n), \quad v(0) = v_0, && (7.7b) \end{aligned}$$

where

$$w(n) = -kd_u(n)(Cz(n) + \mathbf{G}(1)v(n))\,,$$

and

$$z_0 := x_0 - (I-A)^{-1}B(\Phi(u))(0)\,, \quad v_0 := (\Phi(u))(0) - \Phi_r\,.$$

Choose $c \in (k\lambda, K)$ and define

$$\mathbf{H}(z) = -C(zI-A)^{-1}(I-A)^{-1}B + J\,,$$

where $J := 1/c - \mathbf{G}(1)/2$. Then

$$\mathbf{H}(z) = \frac{1}{z-1}(\mathbf{G}(z) - \mathbf{G}(1)) + J\,.$$

Since $c < K$, there exists $\varepsilon > 0$ such that

$$\frac{1}{c} + \mathrm{Re}\,\frac{\mathbf{G}(z)}{z-1} \geq \varepsilon\,, \quad \forall\,|z| > 1\,,$$

and hence, using the identity

$$\mathrm{Re}\left(\frac{1}{e^{i\theta}-1}\right) = -\frac{1}{2}\,, \quad \forall\,\theta \in (0, 2\pi)\,,$$

we may conclude that

$$\mathrm{Re}\,\mathbf{H}(e^{i\theta}) \geq \varepsilon\,, \quad \forall\,\theta \in [0, 2\pi)\,.$$

An application of the discrete-time positive real lemma (see Appendix 7) shows that there exist $P \in L(X)$, $P = P^* \geq 0$, $L \in L(\mathbb{R}, X)$ and $W \in \mathbb{R}$ such that

$$\begin{aligned} A^*PA - P &= -LL^*\,, && (7.8a)\\ A^*P(I-A)^{-1}B &= LW - C^*\,, && (7.8b)\\ W^2 &= 2J - B^*(I-A^*)^{-1}P(I-A)^{-1}B\,. && (7.8c) \end{aligned}$$

For $n \in \mathbb{Z}_+$, define

$$V(n) = \langle z(n), Pz(n)\rangle + \mathbf{G}(1)v(n)^2\,.$$

Using (7.7) and (7.8), we obtain for all $n \in \mathbb{Z}_+$

$$\begin{aligned} &V(n+1) - V(n)\\ &\quad = \langle z(n+1), Pz(n+1)\rangle - \langle z(n), Pz(n)\rangle + \mathbf{G}(1)(v(n+1)^2 - v(n)^2) \end{aligned}$$

$$
\begin{aligned}
&= -(L^*z(n))^2 - 2L^*z(n)Ww(n) + 2Cz(n)w(n) \\
&\qquad + w(n)(2J - W^2)w(n) + \mathbf{G}(1)(w(n)^2 + 2w(n)v(n)) \\
&= -(L^*z(n))^2 - (Ww(n))^2 - 2L^*z(n)Ww(n) \\
&\qquad + 2Cz(n)w(n) + \frac{2}{c}w(n)^2 + 2\mathbf{G}(1)w(n)v(n) \\
&= -(L^*z(n) + Ww(n))^2 + 2Cz(n)w(n) \\
&\qquad + \frac{2}{c}w(n)^2 - 2\mathbf{G}(1)kd_u(n)(\mathbf{G}(1)v(n)^2 + Cz(n)v(n)) \\
&= -(L^*z(n) + Ww(n))^2 + \frac{2}{c}w(n)^2 - 2kd_u(n)(\mathbf{G}(1)v(n) + Cz(n))^2 \\
&= -(L^*z(n) + Ww(n))^2 - 2\left(kd_u(n) - \frac{k^2 d_u(n)^2}{c}\right)(\mathbf{G}(1)v(n) + Cz(n))^2 .
\end{aligned}
$$

Summing then gives

$$
2\sum_{n=0}^{\infty}\left(kd_u(n) - \frac{k^2 d_u(n)^2}{c}\right)(\mathbf{G}(1)v(n) + Cz(n))^2 \le V(0) < \infty . \tag{7.9}
$$

Now, since $c > k\lambda$ and $d_u(n) \in [0, \lambda]$ we have

$$
kd_u(n) - \frac{k^2 d_u(n)^2}{c} = kd_u(n)\left(1 - \frac{kd_u(n)}{c}\right) \ge k\frac{\delta}{\lambda}d_u(n)^2 , \quad \forall\, n \in \mathbb{Z}_+ ,
$$

where $\delta := 1 - k\lambda/c > 0$. Therefore (7.9) implies that

$$
d_u(Cz + \mathbf{G}(1)v) \in l^2(\mathbb{Z}_+, \mathbb{R}) , \tag{7.10}
$$

and hence

$$
w \in l^2(\mathbb{Z}_+, \mathbb{R}) . \tag{7.11}
$$

Appealing to the fact that A is power-stable, we may conclude from (7.7a) and (7.11), by Lemma 7.1, part (2), that

$$
z \in l^2(\mathbb{Z}_+, X) . \tag{7.12}
$$

Consequently, $Cz \in l^2(\mathbb{Z}_+, \mathbb{R})$ and hence, by (7.10) and the boundedness of d_u,

$$
d_u v \in l^2(\mathbb{Z}_+, \mathbb{R}) . \tag{7.13}
$$

From (7.12) and (7.13) we obtain that

$$
(Cz)d_u v \in l^1(\mathbb{Z}_+, \mathbb{R}) . \tag{7.14}
$$

Using (7.9), (7.12)–(7.14) and the boundedness of d it follows that

$$d_u v^2 \in l^1(\mathbb{Z}_+, \mathbb{R}). \tag{7.15}$$

It follows from (7.7b) that, for all $m \in \mathbb{Z}_+$,

$$v(m+1)^2 = v(0)^2 + \sum_{n=0}^{m} w(n)^2 + 2\sum_{n=0}^{m} v(n)w(n). \tag{7.16}$$

Combining (7.16) with (7.11), (7.14) and (7.15) and recalling that $w = -kd_u(Cz + \mathbf{G}(1)v)$, we see that there exists a number $v_\infty \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} v(n)^2 = v_\infty. \tag{7.17}$$

In order to prove statement (1) it is sufficient to show that $v_\infty = 0$. Seeking a contradiction, assume that $v_\infty > 0$. By (7.10), $\lim_{n\to\infty} w(n) = 0$, and thus we may conclude from (7.7b) that

$$\lim_{n\to\infty} (v(n+1) - v(n)) = 0. \tag{7.18}$$

Since $v_\infty > 0$, equations (7.17) and (7.18) yield that $v(n)$ does not change sign for sufficiently large n and so

$$\lim_{n\to\infty} v(n) = \sqrt{v_\infty} \qquad \text{or} \qquad \lim_{n\to\infty} v(n) = -\sqrt{v_\infty}.$$

Assuming that $\lim_{n\to\infty} v(n) = -\sqrt{v_\infty}$ (the case $\lim_{n\to\infty} v(n) = \sqrt{v_\infty}$ can be dealt with in an entirely analogous fashion) we obtain that

$$\Phi_\infty := \lim_{n\to\infty} (\Phi(u))(n) < \Phi_r, \tag{7.19}$$

and thus by Lemma 7.1.2, part (1),

$$\lim_{n\to\infty} x(n) = (I - A)^{-1} B\Phi_\infty. \tag{7.20}$$

It then follows from (7.3b), (7.19) and (7.20) that

$$\lim_{n\to\infty} (u(n+1) - u(n)) = k(r - C(I-A)^{-1}B\Phi_\infty - D\Phi_\infty) = k\mathbf{G}(1)(\Phi_r - \Phi_\infty) > 0.$$

Therefore, $\lim_{n\to\infty} u(n) = \infty$ and u is ultimately non-decreasing, so by (D3) and the assumption that $\Phi_r \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$, we obtain

$$\Phi_\infty = \lim_{n\to\infty} (\Phi(u))(n) = \sup(\mathrm{NVS}\,\Phi) \geq \Phi_r,$$

contradicting (7.19). Therefore, $\lim_{n\to\infty} v(n) = 0$ and consequently

$$\lim_{n\to\infty} (\Phi(u))(n) = \Phi_r ,$$

which is statement (1).

Statement (2) follows from statement (1) and Lemma 7.1.2, part (1). Statement (3) is an easy consequence of statements (1) and (2). Finally, to prove statement (4), let $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$. Then, boundedness of u follows immediately from statement (1) and (D4).

For statement (5), suppose that $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$ is not a critical value of Φ. Therefore, since by statement (4) u is bounded, there exists $d > 0$ and $N > 0$ such that

$$kd_u(n) \in [d, k\lambda] , \quad \forall\, n \geq N . \tag{7.21}$$

Choose $a \in (k\lambda, K)$ and let $\delta \in (0, d)$ be such that $a + \delta < K$. Define

$$\tilde{A} := \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} -(I-A)^{-1}B \\ 1 \end{pmatrix}, \quad \tilde{C} := \begin{pmatrix} C & \mathbf{G}(1) \end{pmatrix},$$

$$\kappa := \frac{a+\delta}{2} \in (k\lambda/2, K/2) , \quad \mathbf{G}_\kappa(z) := \frac{\mathbf{G}(z)}{z-1}\left(1 + \kappa\frac{\mathbf{G}(z)}{z-1}\right)^{-1} , \quad \tilde{A}_\kappa := \tilde{A} - \kappa\tilde{B}\tilde{C} .$$

Since $\frac{\mathbf{G}(z)}{z-1}$ is the transfer function of the system $(\tilde{A}, \tilde{B}, \tilde{C}, 0)$, $\mathbf{G}_\kappa$ is the transfer function of the feedback system $(\tilde{A}_\kappa, \tilde{B}, \tilde{C}, 0)$.

Introduce $\tilde{z} = \begin{pmatrix} z \\ v \end{pmatrix}$; then for all $n \in \mathbb{Z}_+$

$$\tilde{z}(n+1) = \tilde{A}\tilde{z}(n) - kd_u(n)\tilde{B}\tilde{C}\tilde{z}(n) = \tilde{A}_\kappa\tilde{z}(n) - (kd_u(n) - \kappa)\tilde{B}\tilde{C}\tilde{z}(n) . \tag{7.22}$$

To establish the power-stability of $\tilde{A}_\kappa$, we consider, for $(z_0, v_0) \in X \times \mathbb{R}$, the system

$$\begin{pmatrix} z_1(n+1) \\ v_1(n+1) \end{pmatrix} = \tilde{A}_\kappa \begin{pmatrix} z_1(n) \\ v_1(n) \end{pmatrix} , \quad \begin{pmatrix} z_1(0) \\ v_1(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ v_0 \end{pmatrix} , \tag{7.23}$$

which is equivalent to the system

$$z_1(n+1) = Az_1(n) - (I-A)^{-1}Bw_1(n) , \quad z_1(0) = z_0 , \tag{7.24a}$$
$$v_1(n+1) = v_1(n) + w_1(n) , \quad v_1(0) = v_0 , \tag{7.24b}$$

where

$$w_1(n) = -\kappa(Cz_1(n) + \mathbf{G}(1)v_1(n)) . \tag{7.25}$$

Taking the nonlinearity Φ to be the identity and $k = \kappa$ in (7.7), gives the same system as represented in (7.24) and thus by (7.10), $w_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$. An application of Lemma 7.1.2, part (2), to (7.24a) gives $z_1 \in l^2(\mathbb{Z}_+, X)$ and therefore $Cz_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$. Finally, since $w_1, Cz_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$ and using the fact that $\mathbf{G}(1) \neq 0$, we may conclude from (7.25) that $v_1 \in l^2(\mathbb{Z}_+, \mathbb{R})$ and therefore

$$\begin{pmatrix} z_1 \\ v_1 \end{pmatrix} \in l^2(\mathbb{Z}_+, X \times \mathbb{R}) .$$

Since

$$\tilde{A}_\kappa^n \begin{pmatrix} z_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} z_1(n) \\ v_1(n) \end{pmatrix} , \quad \forall n \in \mathbb{Z}_+ ,$$

we have, by a result in [45] (see Proposition 1.2 in [45]), that $\tilde{A}_\kappa$ is power-stable. Since $\tilde{A}_\kappa$ is power-stable

$$\mathbf{G}_\kappa \in H^\infty(\mathbb{E}_1) . \tag{7.26}$$

Moreover, Lemma 2.9 in [27] yields

$$||\mathbf{G}_\kappa||_\infty := \sup_{z \in \mathbb{E}_1} |\mathbf{G}_\kappa(z)| = \frac{1}{\kappa} . \tag{7.27}$$

Setting

$$\gamma := \frac{a - \delta}{2} ; \quad \Psi(f) := -(kd_u - \kappa) f , \quad \forall f \in F(\mathbb{Z}_+, \mathbb{R}) ,$$

and using (7.21), we obtain $|kd_u(n) - \kappa| < \gamma$, for all $n \geq N$ and therefore,

$$|(\Psi(f))(n)| \leq \gamma |f(n)| , \quad \forall f \in F(\mathbb{Z}_+, \mathbb{R}) , \quad \forall n \geq N . \tag{7.28}$$

Clearly, $\kappa > \gamma > 0$, and hence by (7.27)

$$\gamma ||\mathbf{G}_\kappa||_\infty < 1 . \tag{7.29}$$

Let $\rho > 1$ be sufficiently small such that ρA and $\rho \tilde{A}_\kappa$ are power-stable,

$$\mathbf{G}_\kappa \in H^\infty(\mathbb{E}_{1/\rho}) , \tag{7.30}$$

and

$$\gamma \sup_{z \in \mathbb{E}_{1/\rho}} |\mathbf{G}_\kappa(z)| < 1 . \tag{7.31}$$

For all sufficiently small $\rho > 1$, (7.30) follows via a routine argument from (7.26) and the fact that $\mathbf{G} \in H^\infty(\mathbb{E}_{1/\rho})$, whilst (7.31) is a consequence of (7.29) and (7.30) combined with the fact that a holomorphic function which is bounded on a compact set is uniformly continuous on that set.

From (7.22)

$$\tilde{z}(n+1) = \tilde{A}\tilde{z}(n) + \tilde{B}(\Psi(\tilde{C}\tilde{z}))(n). \tag{7.32}$$

Define the bounded operator H from $l^2(\mathbb{Z}_+,\mathbb{R})$ to $l^2(\mathbb{Z}_+,\mathbb{R})$ by setting

$$H(f) = \mathfrak{Z}^{-1}(\mathbf{G}_\kappa \mathfrak{Z}(f)), \quad \forall f \in l^2(\mathbb{Z}_+,\mathbb{R}).$$

By (7.30), H restricts to a bounded operator from $l^2_{1/\rho}(\mathbb{Z}_+,\mathbb{R})$ to $l^2_{1/\rho}(\mathbb{Z}_+,\mathbb{R})$. The $l^2_{1/\rho}(\mathbb{Z}_+,\mathbb{R})$-induced operator norm of H is given by

$$\sup_{z\in\mathbb{E}_{1/\rho}} |\mathbf{G}_\kappa(z)| =: h. \tag{7.33}$$

Noting that

$$\tilde{C}\tilde{z}(n) = \tilde{C}\tilde{A}^n_\kappa \tilde{z}(0) + (H(\Psi(\tilde{C}\tilde{z})))(n),$$

taking the $l^2_{1/\rho}$-norm of $\mathbf{P}^d_n(\tilde{C}\tilde{z})$, using the causality of H and estimating gives

$$\begin{aligned}\left(\sum_{k=0}^{n} |\rho^k \tilde{C}\tilde{z}(k)|^2\right)^{1/2} \quad &\leq \quad \left(\sum_{k=0}^{\infty} |\rho^k \tilde{C}\tilde{A}^k_\kappa \tilde{z}(0)|^2\right)^{1/2} \\ &\quad + h\left(\sum_{k=0}^{n} |\rho^k (\Psi(\tilde{C}\tilde{z}))(k)|^2\right)^{1/2}, \quad \forall n \in \mathbb{Z}_+. \end{aligned} \tag{7.34}$$

Combining (7.28), (7.34) and using the power-stability of $\rho\tilde{A}_\kappa$, we may conclude that there exists $M > 0$ such that

$$\left(\sum_{k=0}^{n} |\rho^k \tilde{C}\tilde{z}(k)|^2\right)^{1/2} \leq M + \gamma h \left(\sum_{k=0}^{n} |\rho^k \tilde{C}\tilde{z}(k)|^2\right)^{1/2}, \quad \forall n \in \mathbb{Z}_+.$$

By (7.31) and (7.33), $\gamma h < 1$, and therefore, $\tilde{C}\tilde{z} \in l^2_{1/\rho}(\mathbb{Z}_+,\mathbb{R})$. Thus

$$w \in l^2_{1/\rho}(\mathbb{Z}_+,\mathbb{R}). \tag{7.35}$$

Define $z_\rho(n) = \rho^n z(n)$ and $w_\rho(n) = \rho^n w(n)$. Then, using (7.7a),

$$z_\rho(n+1) = (\rho A)z_\rho(n) - (I-A)^{-1}B(\rho w_\rho(n)),$$

and therefore since ρw_ρ is a bounded input and ρA is power-stable, z_ρ is bounded. Since w_ρ and z_ρ are bounded, by (7.21), we have that $v_\rho(n) := \rho^n v(n)$ is bounded and so the convergence in statement (1) is of order ρ^{-n}. Now define $x_\rho(n) = \rho^n(x(n) - (I-A)^{-1}B\Phi_r)$, then $x_\rho(n+1) = (\rho A)x_\rho(n) + B(\rho v_\rho(n))$. Therefore, since ρv_ρ is a bounded input and ρA is power-stable, x_ρ is bounded. Thus the convergence in statement (2) is of order ρ^{-n}.

Finally, for statement (6), since $y - r \in l^2_{1/\rho}(\mathbb{Z}_+, \mathbb{R})$, $u(\cdot + 1) - u(\cdot) \in l^1(\mathbb{Z}_+, \mathbb{R})$ and so u converges to a finite number. $\square$

We see from the proof of Theorem 7.1.4 that (D4) is only needed for statement (4). One of the conditions imposed in Theorem 7.1.4 is that $r/\mathbf{G}(1) \in \operatorname{clos}(\operatorname{NVS}\Phi)$. The following proposition shows that this condition is close to being necessary for tracking insofar as, if tracking of r is achievable whilst maintaining boundedness of $\Phi(u)$, then $r/\mathbf{G}(1) \in \operatorname{clos}(\operatorname{NVS}\Phi)$.

Proposition 7.1.5 *Let $\lambda > 0$ and $r \in \mathbb{R}$. Suppose that $\Phi \in \mathcal{N}_d(\lambda)$, A is power-stable and $\mathbf{G}(1) > 0$. If there exist an initial condition $x_0 \in X$ and a function $u \in F(\mathbb{Z}_+, \mathbb{R})$ such that $\Phi(u)$ is bounded and*

$$\lim_{n\to\infty} [Cx(n) + D(\Phi(u))(n)] = r\,,$$

where $x \in F(\mathbb{Z}_+, X)$ is given by (7.2a), then $r/\mathbf{G}(1) \in \operatorname{clos}(\operatorname{NVS}\Phi)$.

Proof: Since $\Phi(u)$ is bounded and A is power-stable, x is bounded. Let $n \in \mathbb{Z}_+$ and define $y : \mathbb{Z}_+ \to \mathbb{R}$ by (7.2b), then

$$y(n) = C(x(n) - (I - A)^{-1}B(\Phi(u))(n)) + \mathbf{G}(1)(\Phi(u))(n)\,,$$

and therefore

$$C(A - I)^{-1}(x(n + 1) - x(n)) = y(n) - \mathbf{G}(1)(\Phi(u))(n)\,.$$

For $p, m \in \mathbb{Z}_+$ with $p > m$, summing the above from m to $p - 1$ gives

$$C(A - I)^{-1}(x(p) - x(m)) = \sum_{k=m}^{p-1}(y(k) - \mathbf{G}(1)(\Phi(u))(k))\,. \tag{7.36}$$

Seeking a contradiction, let us suppose that $r/\mathbf{G}(1) \notin \operatorname{clos}(\operatorname{NVS}\Phi)$. Since $\lim_{n\to\infty} y(n) = r$ and $\operatorname{clos}(\operatorname{NVS}\Phi)$ is an interval (see Remark 5.3.1, part (3)), there exist $\varepsilon > 0$, $\beta \in \{-1, 1\}$ and $m \in \mathbb{Z}_+$ such that

$$\beta(y(n) - \mathbf{G}(1)(\Phi(u))(n)) \geq \varepsilon\,, \quad \forall\, n \geq m\,.$$

Combining the above with (7.36), it follows that

$$\beta C(A-I)^{-1}(x(n)-x(m)) = \sum_{k=m}^{n-1} \beta(y(k)-\mathbf{G}(1)(\Phi(u))(k)) \geq \varepsilon(n-m)\,, \quad \forall\, n > m\,.$$

Therefore $\lim_{n\to\infty} \beta C(A - I)^{-1}x(n) = \infty$, contradicting the boundedness of x. $\square$

7.2 Example: finite-dimensional system

As a simple example we consider

$$\begin{aligned} x(n+1) &= \begin{pmatrix} 0.5 & 4 \\ 0 & -0.4 \end{pmatrix} x(n) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} (\mathcal{B}^d_{0.5,5}(u))(n)\,, \quad x(0) = 0\,, \\ y(n) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(n)\,, \end{aligned}$$

where $\mathcal{B}^d_{0.5,5}$ is the discrete-time backlash operator defined in (4.44). The transfer function is given by

$$\mathbf{G}(z) = \frac{2z + 4.8}{(z-0.5)(z+0.4)}\,.$$

Now $\mathcal{B}^d_{0.5,5} \in \mathcal{N}_d(1)$, NVS $\mathcal{B}^d_{0.5,5} = \mathbb{R}$ and using Corollary 7.1.3 we get the following lower bound for K

$$K \geq \frac{1}{\|\mathbf{E}\|_\infty + \mathbf{G}(1)/2} \approx 0.0355\,.$$

For $r = 1$, we have

$$\Phi_r = \frac{r}{\mathbf{G}(1)} = \frac{7}{68} \in \operatorname{int}(\mathrm{NVS}\,\mathcal{B}^d_{0.5,5})\,.$$

Therefore, by Theorem 7.1.4, for any $k \in (0, 0.0355)$, the integral control, $u(n+1) = u(n) + k[r - y(n)]$, with $u(0) = 0$, guarantees asymptotic tracking. For purposes of illustration we choose $k = 0.035$.

Figure 29 depicts the output behaviour of the system under integral control, Figure 30 depicts the corresponding control input, Figure 31 shows the input of the backlash operator and Figure 32 shows the state vector. We remark that since $r/\mathbf{G}(1)$ is a critical numerical value of $\mathcal{B}_{0.5,5}$, exponential convergence is not guaranteed by Theorem 7.1.4 and indeed convergence seems to be slow. Also Theorem 7.1.4 does not guarantee the convergence of u, and indeed it appears that u does not converge.

Figures 29–32 were generated using SIMULINK Simulation Software within MATLAB.

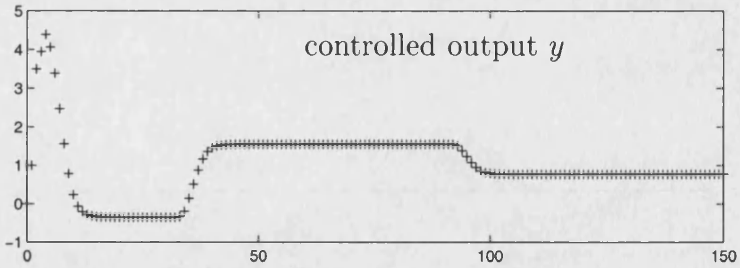

Figure 29: Controlled output

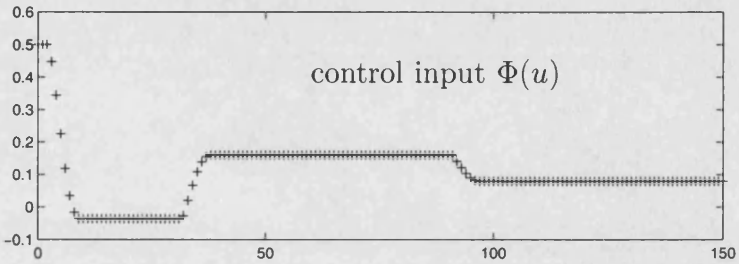

Figure 30: Control input

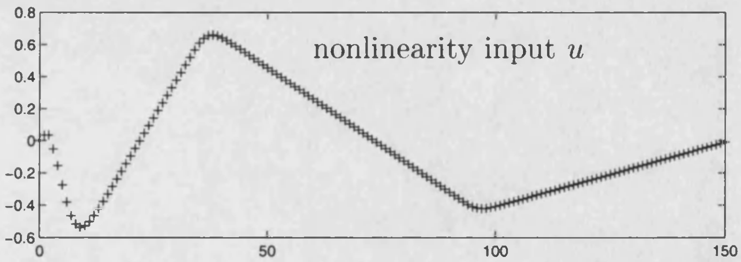

Figure 31: Input of backlash operator

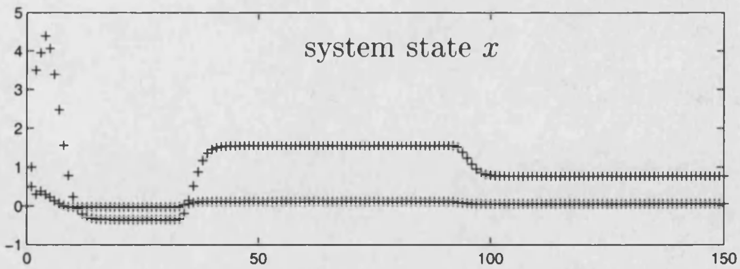

Figure 32: System state

7.3 Notes and references

We remark that Theorem 7.1.4 (the discrete-time counterpart of the continuous-time result expressed in Theorem 6.1.2) is new for both static nonlinearities and for finite-dimensional linear systems. Lemma 7.1.2 and Corollary 7.1.3 are discrete-time counterparts of results in [25] (see Lemma 3.3 and Corollary 3.4 in [25]). Proposition 7.1.5 is an extension of a result in [23] (see Proposition 2.3 in [23]).

Chapter 8

Sampled-data low-gain control of regular linear systems subject to input hysteresis

8.1 Sampled-data integral control in the presence of input nonlinearities in $\mathcal{N}_{sd}(\lambda)$

The aim in this chapter is to show that for an exponentially stable, regular, linear, infinite-dimensional, continuous-time, single-input, single-output system with transfer function $\mathbf{G}(s)$, subject to a continuous-time dynamic input nonlinearity $\tilde{\Phi}$, the output $y(t)$ of the sampled-data closed-loop system, shown in Figure 5, converges to the reference value r as $t \to \infty$, provided that $\mathbf{G}(0) > 0$, r is feasible in some natural sense and $k > 0$ is sufficiently small.

Consider a single-input, single-output, discrete-time system

$$x^d(n+1) = A^d x^d(n) + B^d u^d(n)\,, \quad x(0) = x_0 \in X\,, \tag{8.1a}$$
$$y^d(n) = C^d x^d(n) + D^d u^d(n)\,, \tag{8.1b}$$

evolving on a real Hilbert space X. The transfer function $\mathbf{G}^d$ of (8.1) is given by

$$\mathbf{G}^d(z) = C^d(zI - A^d)^{-1}B^d + D^d\,.$$

As in Chapter 7, if $\mathbf{G}^d \in H^\infty(\mathbb{E}_\alpha)$ for some $\alpha \in (0,1)$ (which is the case if A^d is power-stable) and $\mathbf{G}^d(1) > 0$, then

$$1 + k\,\mathrm{Re}\,\frac{\mathbf{G}^d(z)}{z-1} \geq 0\,, \quad \forall\, z \in \mathbb{E}_1\,, \tag{8.2}$$

for all sufficiently small $k > 0$, see Lemma 2.9 in [27]. We define

$$K := \sup\{k > 0 \,|\, (8.2) \text{ holds}\}\,. \tag{8.3}$$

We recall the discrete-time closed-loop system considered in Chapter 7

$$x^d(n+1) = A^d x^d(n) + B^d(\Phi^d(u^d))(n), \;\; x^d(0) = x_0 \in X, \tag{8.4a}$$
$$u^d(n+1) = u^d(n) + k(r - C^d x^d(n) - D^d(\Phi^d(u^d))(n)), \;\; u^d(0) = u_0 \in \mathbb{R}, \tag{8.4b}$$

where k is a real parameter and $\Phi^d \in \mathcal{N}_d(\lambda)$.

Let $u^d \in F(\mathbb{Z}_+, \mathbb{R})$ and apply the continuous-time signal

$$u = H_\tau u^d\,, \tag{8.5}$$

(where H_τ is the standard hold operator defined in Chapter 4) to the continuous-time system given by (3.7) (where $(A, B, C, D) \in \mathcal{L}$). Then the state $x(n\tau + t)$ satisfies

$$x(n\tau + t) = \mathbf{T}_t x(n\tau) + (\mathbf{T}_t - I)A^{-1}Bu^d(n)\,, \quad \forall\, n \in \mathbb{Z}_+, \quad \forall\, t \in [0, \tau)\,.$$

Accordingly, we define $x^d : \mathbb{Z}_+ \to X$ by

$$x^d(n) = x(n\tau)\,. \tag{8.6}$$

Clearly, $\mathbf{T}_\tau \in L(X)$ and $(\mathbf{T}_\tau - I)A^{-1}B \in L(\mathbb{R}, X)$ define appropriate state-space operators for the state evolution of the discretization of (3.7a). However, in general, regularity only guarantees that $y \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ so that, even with piecewise constant input functions, standard sampling of the output is not defined. Moreover, even if the output function is continuous (in which case standard sampling is defined), in general the resulting discrete-time system will not have a bounded observation operator. We therefore distinguish two cases: bounded and unbounded observation.

Bounded observation

Assume that $C = C_L \in L(X, \mathbb{R})$. If $x_0 \in X$ and u is given by (8.5), then the output y given by (3.7b) is piecewise continuous, the discontinuities being at $n\tau$. It is clear that y is right-continuous at $n\tau$ for all $n \in \mathbb{Z}_+$. We define

$$y^d := S_\tau y \tag{8.7}$$

(where S_τ is the standard sampling operator defined in Section 4.5) and

$$\begin{pmatrix} A^d & B^d \\ C^d & D^d \end{pmatrix} := \begin{pmatrix} \mathbf{T}_\tau & (\mathbf{T}_\tau - I)A^{-1}B \\ C & D \end{pmatrix}. \tag{8.8}$$

The proof of the following proposition is an immediate consequence of Proposition 4.1 in [27].

Proposition 8.1.1 *Suppose that $\mathbf{T}_t$ is exponentially stable and that the observation operator C is bounded. Let $\tau > 0$ and $u^d \in F(\mathbb{Z}_+, \mathbb{R})$. If u given by (8.5) is applied to (3.7), then x^d and y^d given by (8.6) and (8.7), respectively, satisfy (8.1) where (A^d, B^d, C^d, D^d) is given by (8.8). Moreover, A^d is power-stable and we have that*

$$\mathbf{G}^d(1) = C^d(I - A^d)^{-1}B^d + D^d = \mathbf{G}(0). \tag{8.9}$$

For $\Phi \in \mathcal{N}_{sd}(\lambda)$, let $\tilde{\Phi}$ denote the extension of Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ given by (4.31) and let Φ^d denote the discretization of $\tilde{\Phi}$ given by (4.39). For $u \in \mathcal{S}_\tau$, we have $H_\tau S_\tau u = u$ and so, by Lemma 4.4.5, part (4)

$$\tilde{\Phi}\, H_\tau = H_\tau\, \Phi^d. \tag{8.10}$$

Consider the continuous-time system (3.7) with continuous-time input nonlinearity $\tilde{\Phi}$ and $(A, B, C, D) \in \mathcal{L}$

$$\dot{x} = Ax + B\tilde{\Phi}(u), \quad x(0) = x_0 \in X, \tag{8.11a}$$

$$y = C_L x + D\tilde{\Phi}(u), \tag{8.11b}$$

controlled by the sampled-data integrator

$$u(t) = u^d(n), \quad \text{for } t \in [n\tau, (n+1)\tau),\ n \in \mathbb{Z}_+, \tag{8.12a}$$

$$y^d(n) = y(n\tau), \quad n \in \mathbb{Z}_+, \tag{8.12b}$$

$$u^d(n+1) = u^d(n) + k(r - y^d(n)), \quad u^d(0) = u_0 \in \mathbb{R},\ n \in \mathbb{Z}_+. \tag{8.12c}$$

Theorem 8.1.2 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_{sd}(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, C is bounded and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in$ clos (NVS Φ). Let $K > 0$ be defined by (8.3), where (A^d, B^d, C^d, D^d) is given by (8.8). Then, for all $k \in (0, K/\lambda)$ and all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of the closed-loop system given by (8.11) and (8.12) satisfies*

(1) $\lim_{t\to\infty}(\tilde{\Phi}(u))(t) = \Phi_r$,

(2) $\lim_{t\to\infty}\|x(t) + A^{-1}B\Phi_r\| = 0$,

(3) $\lim_{t\to\infty} y(t) = r$,

(4) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$*, then* u *is bounded,*

(5) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ *and* Φ_r *is not a critical numerical value of* Φ*, then the convergence in (1), (2) and (3) is of order* $\exp(-\rho t)$ *for some* $\rho > 0$,

(6) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ *and* Φ_r *is not a critical numerical value of* Φ*, then there exists* $u_\infty \in \mathbb{R}$ *such that* $\lim_{t\to\infty} u(t) = u_\infty$.

Proof: Let $(x(\cdot), u(\cdot))$ be the unique solution of the closed-loop system given by (8.11) and (8.12). Let Φ^d be given by (4.39) and so $\Phi^d \in \mathcal{N}_d(\lambda)$ (by Proposition 5.3.5) and

$$(\tilde{\Phi}(u))(n\tau) = (\tilde{\Phi}(H_\tau u^d))(n\tau) = (\Phi^d(u^d))(n)\,, \quad \forall\, n \in \mathbb{Z}_+ .$$

Note that by Proposition 4.5.6

$$\mathrm{NVS}\,\Phi^d = \mathrm{NVS}\,\Phi\,. \tag{8.13}$$

Defining $x^d \in F(\mathbb{Z}_+, \mathbb{R})$ by (8.6), it follows from Proposition 8.1.1 that (x^d, u^d) satisfies the discrete-time closed-loop system (8.4), where (A^d, B^d, C^d, D^d) is given by (8.8). Therefore, using Theorem 7.1.4, Proposition 8.1.1 and (8.13) we see that for all $k \in (0, K/\lambda)$

$$\lim_{n\to\infty} (\Phi^d(u^d))(n) = \Phi_r\,. \tag{8.14}$$

This implies that for all $k \in (0, K/\lambda)$, $\lim_{t\to\infty}(H_\tau(\Phi^d(u^d)))(t) = \Phi_r$ and so by (8.10), $\lim_{t\to\infty}(\tilde{\Phi}(u))(t) = \Phi_r$, which is statement (1). Statement (2) is a consequence of statement (1) and Lemma 3.1.4. Statement (3) follows easily from statements (1) and (2) and the boundedness of C. To prove statement (4), assume that $\Phi_r \in \operatorname{int}\mathrm{NVS}\,\Phi$. Then, by (8.13), $\Phi_r \in \operatorname{int}\mathrm{NVS}\,\Phi^d$. Boundedness of u^d and thus boundedness of u now follows immediately from (8.14) and the fact that (D4) holds for Φ^d.

For statements (5) and (6) assume that $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and that Φ_r is not a critical numerical value of Φ. Then by Proposition 5.3.7, Φ_r is not a critical numerical value of Φ^d. So by Theorem 7.1.4, part (5), there exists $\xi > 1$ such that $\lim_{n\to\infty} \xi^n((\tilde{\Phi}(u))(n\tau) - \Phi_r) = 0$ and $\lim_{n\to\infty} \xi^n \|x(n\tau) + A^{-1}B\Phi_r\| = 0$. Therefore, for $\rho := (\ln \xi)/\tau > 0$, we have, using (8.10), $\lim_{t\to\infty} \exp(\rho t)((\tilde{\Phi}(u))(t) - \Phi_r) = 0$ and $\lim_{t\to\infty} \exp(\rho t)\|x(t) + A^{-1}B\Phi_r\| = 0$. Since C is bounded, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for some $\alpha < 0$ (see Lemma 2.3 in [24]) and therefore the fact that the convergence in (3) is of order $\exp(-\rho t)$, for some $\rho > 0$, follows as in the proof of Theorem 6.1.2. By Theorem 7.1.4, part (6), u^d converges, and so u converges. $\square$

Let $\sigma(\cdot)$ denote the *step response* of the regular system $(A, B, C, D) \in \mathfrak{L}$, i.e.

$$\sigma(t) = C_L \int_0^t \mathbf{T}_{t-\tau} BU(\tau)\, d\tau + DU(t)\,.$$

Define the *step-response error* $\varepsilon(\cdot)$ by

$$\varepsilon(t) = \sigma(t) - \mathbf{G}(0)\,,$$

with Laplace transforms given by $[\mathfrak{L}(\sigma)](s) = \mathbf{G}(s)/s$ and $[\mathfrak{L}(\varepsilon)](s) = (\mathbf{G}(s) - \mathbf{G}(0))/s$, respectively.

The following proposition is due to N. Özdemir and S. Townley [33] (see Remark 3.7 in [33]). For completeness we include a proof.

Proposition 8.1.3 *Assume that $(A, B, C, D) \in \mathfrak{L}$ and C is bounded. Let $K > 0$ be defined by (8.3), where (A^d, B^d, C^d, D^d) is given by (8.8). Then*

$$K \geq \frac{1}{\sum_{k=0}^{\infty} |\varepsilon(k\tau)| + \mathbf{G}(0)/2}\,, \tag{8.15}$$

and if $D = 0$ and $\sigma(\cdot)$ is non-decreasing, then $\mathbf{G}'(0) \leq 0$ and

$$K \geq \frac{1}{|\mathbf{G}'(0)|/\tau + 3\mathbf{G}(0)/2}\,. \tag{8.16}$$

Proof: Defining $\tilde{\sigma}(z) = \sum_{k=0}^{\infty} \sigma(k\tau) z^{-k}$, we may write $\mathbf{G}^d(z) = (1 - z^{-1})\tilde{\sigma}(z)$. Set

$$\mathbf{E}^d(z) := \frac{\mathbf{G}^d(z) - \mathbf{G}^d(1)}{z - 1}\,,$$

then, using (8.9)

$$\mathbf{E}^d(z) = \frac{1}{z}\left(\tilde{\sigma}(z) - \frac{\mathbf{G}(0) z}{z - 1}\right)\,.$$

Therefore

$$\|\mathbf{E}^d\|_\infty = \sup_{\theta \in [0, 2\pi)} |\mathbf{E}^d(e^{i\theta})| = \sup_{\theta \in [0, 2\pi)} \Big|\sum_{k=0}^{\infty} \varepsilon(k\tau) e^{-ik\theta}\Big| \leq \sum_{k=0}^{\infty} |\varepsilon(k\tau)|\,.$$

Combining the above with Corollary 7.1.3 gives (8.15). Since C is bounded, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ (see Lemma 2.3 in [24]) and therefore σ is continuous. If $D = 0$ and σ is non-decreasing, then $\varepsilon(t) \leq 0$ for all $t \in \mathbb{R}_+$. Therefore, defining $\mathbf{E}(s) := (\mathbf{G}(s) - \mathbf{G}(0))/s$, we have for $s \in \mathbb{C}_0$

$$-\mathbf{G}'(0) = -\mathbf{E}(0) = -\int_0^\infty \varepsilon(\tau)\, d\tau = \int_0^\infty |\varepsilon(\tau)| \geq 0\,,$$

and thus $\mathbf{G}'(0) \le 0$. Additionally

$$\begin{aligned}\sum_{k=0}^{\infty} |\varepsilon(k\tau)| &= -\sum_{k=0}^{\infty} \varepsilon(k\tau) = -\sum_{k=1}^{\infty} \varepsilon(k\tau) - \varepsilon(0) \le -\frac{1}{\tau}\int_0^{\infty} \varepsilon(t)\,dt + \mathbf{G}(0) \\ &= -\frac{1}{\tau}(\mathfrak{L}(\varepsilon))(0) + \mathbf{G}(0) = -\frac{1}{\tau}\mathbf{G}'(0) + \mathbf{G}(0)\,.\end{aligned}$$

Combining the above with (8.15) gives (8.16). □

Unbounded observation

As mentioned earlier, in this case we cannot define a sampled output via (8.7). Instead, we introduce a generalized sampling operation. In the following, let $w \in L^2([0,\tau],\mathbb{R})$ be a function satisfying the conditions

$$\text{(a)} \quad \int_0^{\tau} w(t)\,dt = 1 \quad \text{and} \quad \text{(b)} \quad \int_0^{\tau} w(t)\mathbf{T}_t x\,dt \in X_1 \quad \forall x \in X\,. \tag{8.17}$$

Whilst condition (8.17)(b) is difficult to check for general w, it is easy to show (using integration by parts) that (8.17)(b) holds if there exists a partition $0 = t_0 < t_1 < \ldots < t_m = \tau$ such that $w|_{(t_{i-1},t_i)} \in W^{1,1}((t_{i-1},t_i),\mathbb{R})$ for $i = 1,2,\ldots,m$. We define a generalized sampling operation by

$$y^d(n) = \int_0^{\tau} w(t)y(n\tau + t)\,dt\,, \quad \forall n \in \mathbb{Z}_+\,. \tag{8.18}$$

Introducing the linear operator

$$L : X \to X_1\,, \quad x \mapsto \int_0^{\tau} w(t)\mathbf{T}_t x\,dt\,,$$

we define

$$\begin{pmatrix} A^d & B^d \\ C^d & D^d \end{pmatrix} := \begin{pmatrix} \mathbf{T}_\tau & (\mathbf{T}_\tau - I)A^{-1}B \\ CL & CLA^{-1}B + \mathbf{G}(0) \end{pmatrix}. \tag{8.19}$$

The following result is an immediate conquence of Proposition 3.4 in [23].

Proposition 8.1.4 *Suppose that $\mathbf{T}_t$ is exponentially stable. Let $\tau > 0$ and $u^d \in F(\mathbb{Z}_+,\mathbb{R})$. If u given by (8.5) is applied to (3.7), then x^d and y^d given by (8.6) and (8.18), respectively, satisfy (8.1), where (A^d,B^d,C^d,D^d) is given by (8.19). Moreover, A^d is power-stable, $C^d \in L(X,\mathbb{R})$ and (8.9) is satisfied.*

Consider the following sampled-data low-gain controller for (8.11)

$$
\begin{aligned}
u(t) &= u^d(n), \quad \text{for } t \in [n\tau, (n+1)\tau),\ n \in \mathbb{Z}_+, && \text{(8.20a)}\\
y^d(n) &= \int_0^\tau w(t) y(n\tau + t)\, dt, \quad n \in \mathbb{Z}_+, && \text{(8.20b)}\\
u^d(n+1) &= u^d(n) + k(r - y^d(n)), \quad u(0) = u_0 \in \mathbb{R},\ n \in \mathbb{Z}_+. && \text{(8.20c)}
\end{aligned}
$$

Theorem 8.1.5 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_{sd}(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$. Let $K > 0$ be defined by (8.3), where (A^d, B^d, C^d, D^d) is given by (8.19). Then, for all $k \in (0, K/\lambda)$ and all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of the closed-loop system given by (8.11) and (8.20) satisfies*

(1) $\lim_{t\to\infty}(\tilde{\Phi}(u))(t) = \Phi_r$*,*

(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\Phi_r\| = 0$*,*

(3) $\lim_{t\to\infty}[r - y(t) + C_L \mathbf{T}_t x_0] = 0$*,*

(4) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$, then u is bounded,

(5) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and Φ_r is not a critical numerical value of Φ, then the convergence in (1) and (2) is of order $\exp(-\rho t)$ for some $\rho > 0$, moreover, if $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f^\alpha(\mathbb{R}_+)$ for some $\alpha < 0$, then the convergence in (3) is of order $\exp(-\rho t)$ for some $\rho \in (0, -\alpha)$,

(6) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and Φ_r is not a critical numerical value of Φ, then there exists $u_\infty \in \mathbb{R}$ such that $\lim_{t\to\infty} u(t) = u_\infty$.

Proof: By using Proposition 8.1.4 instead of Proposition 8.1.1, statements (1), (2) and (4) follow exactly as in the proof of Theorem 8.1.2. Statement (3) follows from statement (1) as in the proof of Theorem 6.1.2. For statements (5) and (6) assume that $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and that Φ_r is not a critical numerical value of Φ. Then the fact that the convergence in statements (1) and (2) is of order $\exp(-\rho t)$ for some $\rho > 0$ and that u converges, follows as in the proof of Theorem 8.1.2. Let us now assume that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f^\alpha(\mathbb{R}_+)$ for some $\alpha < 0$, then the fact that the convergence in (3) is of order $\exp(-\rho t)$ for some $\rho \in (0, -\alpha)$ follows as in the proof of Theorem 6.1.2. □

Remark 8.1.6 We see from the proofs of Theorems 8.1.2 and 8.1.5, and the results related to extending $\Phi \in \mathcal{N}_{sd}(\lambda)$ to $\mathrm{NPC}_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$, that Φ need not satisfy (C5) but only the weaker assumption (C5′). ◇

8.2 Example: controlled diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $[0,1]$, with scalar nonlinear pointwise control action (applied at point $x_1 \in (0,1)$, via an operator $\Phi \in \mathcal{N}_{sd}(\lambda)$) and delayed (delay $T \geq 0$) scalar observation generated by a spatial averaging of the delayed state over an ε-neighbourhood of a point $x_2 \in (x_1, 1)$, where $\varepsilon \in (0, \min(1-x_2, x_2-x_1))$.

We formally write this single-input, single-output system as

$$\begin{aligned} z_t(t,x) &= \kappa z_{xx}(t,x) + \delta(x-x_1)(\tilde{\Phi}(u))(t)\,, \\ y(t) &= \frac{1}{2\varepsilon}\int_{x_2-\varepsilon}^{x_2+\varepsilon} z(t-T,x)\,dx\,, \end{aligned}$$

with boundary conditions

$$z(t,0) = 0 = z(t,1)\,, \quad \forall\, t > 0\,.$$

For simplicity, we assume zero initial conditions

$$z(t,x) = 0\,, \quad \forall\,(t,x) \in [-T,0] \times [0,1]\,.$$

With input $(\tilde{\Phi}(u))(\cdot)$ and output $y(\cdot)$, this example qualifies as a regular linear system with bounded observation and with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sT}\sinh\left(x_1\sqrt{\frac{s}{\kappa}}\right)\left[\cosh\left((1-x_2+\varepsilon)\sqrt{\frac{s}{\kappa}}\right) - \cosh\left((1-x_2-\varepsilon)\sqrt{\frac{s}{\kappa}}\right)\right]}{2\varepsilon s \sinh\sqrt{\frac{s}{\kappa}}}\,.$$

Since the observation is bounded, we may sample the output using the standard sampling operation given by (8.7). Further analysis (invoking application of the maximum principle for the heat equation which, for brevity, we omit here) confirms the physical intuition that the impulse response $\mathfrak{L}^{-1}(\mathbf{G})$ is nonnegative valued. Consequently, the corresponding step-response is non-decreasing, and therefore we may apply Proposition 8.1.3 to obtain the following lower bound for K

$$K \geq \frac{1}{|\mathbf{G}'(0)|/\tau + 3\mathbf{G}(0)/2} =: K_L\,. \tag{8.21}$$

A simple calculation yields that

$$\mathbf{G}(0) = \frac{x_1(1-x_2)}{\kappa}\,, \quad \mathbf{G}'(0) = -\frac{x_1(1-x_2)(6T\kappa + 1 - \varepsilon^2 - x_1^2 - (1-x_2)^2)}{6\kappa^2}\,,$$

and therefore, using (8.21)

$$K \geq K_L = \frac{6\kappa^2\tau}{x_1(1-x_2)(6T\kappa + 1 - \varepsilon^2 - x_1^2 - (1-x_2)^2 + 9\kappa\tau)}.$$

By Theorem 8.1.2, for all $k \in (0, K_L/\lambda) \subset (0, K/\lambda)$, the sampled-data control (8.12), guarantees asymptotic tracking of all reference values r which are feasible in the sense that $r/\mathbf{G}(0) \in \operatorname{clos}(\operatorname{NVS}\Phi)$. For purposes of illustration, we adopt the following values

$$\kappa = 0.1, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad T = 1, \quad \tau = 0.5, \quad \varepsilon = 0.01.$$

For these specific values we obtain $K_L \approx 0.147$.

We consider relay and backlash hysteresis operators:

(a) Let $\Phi = \mathcal{R}_\xi$ be a relay hysteresis operator, defined in (4.12), where $\xi = 0$, $a_1 = -1$, $a_2 = 1$, $\rho_1(v) = \sqrt{v + 1.1}$ and $\rho_2(v) = \sqrt{0.1} + \sqrt{2.1} - \sqrt{1.1 - v}$. Then $\Phi \in \mathcal{C}(1.6)$ and $\operatorname{NVS}\Phi = \operatorname{im}\rho_1 \cup \operatorname{im}\rho_2 = \mathbb{R}$. For reference value $r = 1.54$

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 1.386 \in \operatorname{int}(\operatorname{NVS}\Phi).$$

In each of the following three cases of admissible controller gain

(i) $k = 0.08$, (ii) $k = 0.06$, (iii) $k = 0.04$,

Figure 33 depicts the output behaviour of the system under sampled-data control, Figure 34 depicts the corresponding control input and Figure 35 shows the input of the relay hysteresis operator. Since Φ_r is not a critical value of Φ, statements (5) and (6) of Theorem 8.1.2 hold and therefore the convergence seen in Figures 33 and 34 is of exponential order and u is seen to converge in Figure 35. We see from Figure 35 that for (i), $\lim_{t\to\infty} u(t) = \rho_1^{-1}(\Phi_r)$ and for (ii) and (iii), $\lim_{t\to\infty} u(t) = \rho_2^{-1}(\Phi_r)$.

(b) Let $\Phi = \mathcal{B}_{0.5,0}$ be a standard backlash hysteresis operator as defined in Section 4.3. Then $\Phi \in \mathcal{C}(1)$ and $\operatorname{NVS}\Phi = \mathbb{R}$. For reference value $r = 1$

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 0.9 \in \operatorname{int}(\operatorname{NVS}\Phi).$$

In each of the following three cases of admissible controller gain

(i) $k = 0.145$ (solid), (ii) $k = 0.11$ (dashdot), (iii) $k = 0.08$ (dotted),

Figure 36 depicts the output behaviour of the system under sampled-data control,

Figure 37 depicts the corresponding control input and Figure 38 shows the input of the backlash operator. We remark that the convergence of $u(t)$ as $t \to \infty$ is not guaranteed by Theorem 8.1.2 and in fact it seems that u does not converge in all three cases.

Figures 33–38 were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

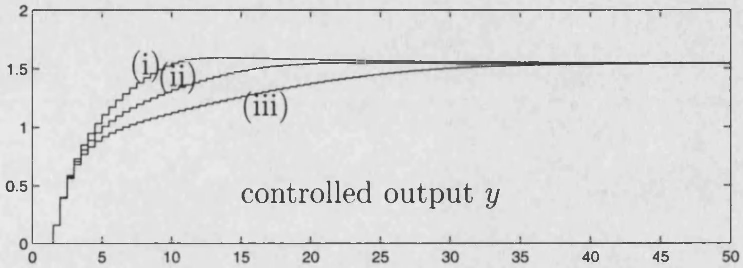

Figure 33: Controlled output

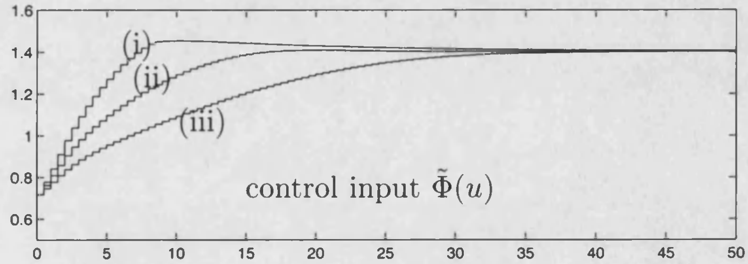

Figure 34: Control input

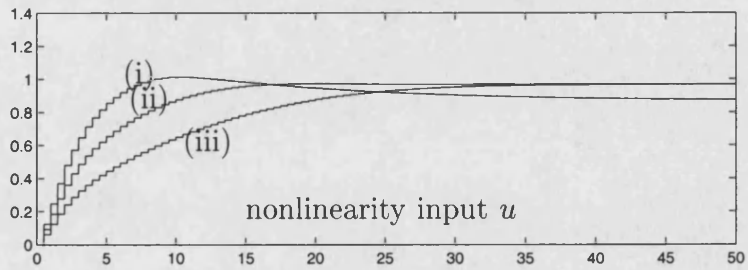

Figure 35: Input of relay operator

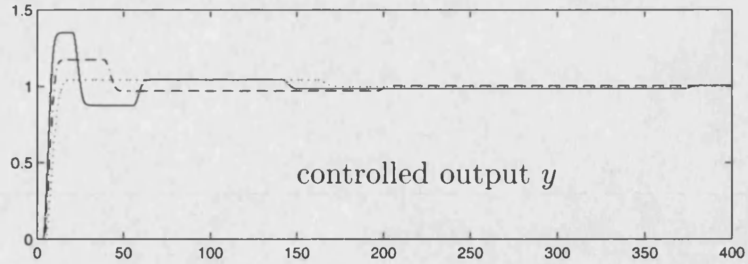

Figure 36: Controlled output

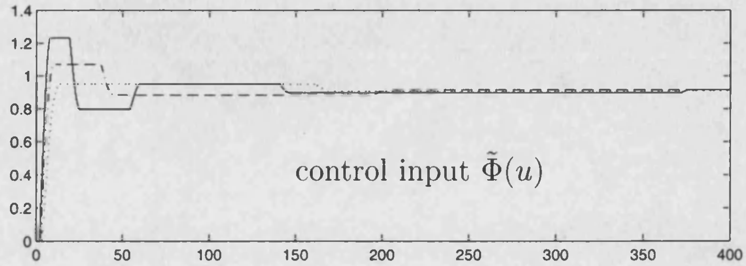

Figure 37: Control input

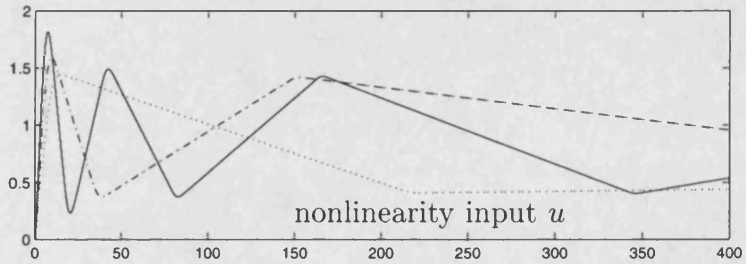

Figure 38: Input of backlash operator

8.3 Notes and references

Theorems 8.1.2 and 8.1.5 are new for both finite-dimensional plants and static nonlinearities. The proof of Proposition 8.1.3 was first seen in [33] (see Remark 3.7 in [33]). Statements (1)–(4) of Theorems 8.1.2 and 8.1.5 can be found in [21] by Logemann and Mawby.

Chapter 9

Time-varying and adaptive integral control of linear systems subject to input hysteresis

In Chapter 5, constant-gain integral control was considered in the context of systems $(A, B, C, D) \in \mathcal{L}$ with input nonlinearities $\Phi \in \mathcal{N}_c(\lambda)$: there, the existence of a value $k^* > 0$, with the property that asymptotic tracking of "feasible" reference signals r is ensured for all fixed gains $k \in (0, k^*)$, is established. However, k^* is, in general, a function of the plant data and so, in the presence of plant uncertainty, may fail to be computable in practice. In such circumstances, one might be led naïvely to consider a time-dependent gain strategy $t \mapsto k(t) > 0$ with $k(t)$ approaching zero as t tends to infinity. We consider this situation in Section 9.1. In Section 9.2, we consider an adaptive gain strategy, where $k(t) > 0$ is updated on the basis of output information from the plant. Section 9.3 contains the discrete-time and sampled-data analogies of the continuous-time result contained in Section 9.2.

9.1 Continuous-time integral control with time-varying gain in the presence of input nonlinearities in $\mathcal{N}_c(\lambda)$

Let $\lambda > 0$, $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$ and $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$. We denote, by $r \in \mathbb{R}$, the value of the constant reference signal to be tracked by the output $y(t)$.

We will investigate integral control action

$$u(t) = u_0 + \int_0^t k(\tau)[r - C_L x(\tau) - D(\Phi(u))(\tau)]\, d\tau\,,$$

with time-varying gain $k(\cdot)$, leading to the following nonlinear system of differential equations

$$\dot{x}(t) = Ax(t) + B(\Phi(u))(t)\,, \quad x(0) = x_0 \in X\,, \tag{9.1a}$$

$$\dot{u}(t) = k(t)[r - C_L x(t) - D(\Phi(u))(t)]\,, \quad u(0) = u_0 \in \mathbb{R}\,. \tag{9.1b}$$

For $a \in (0, \infty]$, a continuous function

$$[0, a) \to X \times \mathbb{R}, \quad t \mapsto (x(t), u(t))$$

is a *solution* of (9.1) if $(x(\cdot), u(\cdot))$ is absolutely continuous as a $(X_{-1} \times \mathbb{R})$-valued function, $x(t) \in \mathrm{dom}(C_L)$ for almost all $t \in [0, a)$, $(x(0), u(0)) = (x_0, u_0)$ and the differential equations in (9.1) are satisfied almost everywhere on $[0, a)$, where the derivative in (9.1a) should be interpreted in the space X_{-1}.†

An application of a well-known result on abstract Cauchy problems (see Pazy [34], Theorem 2.4, p. 107) shows that a continuous $(X \times \mathbb{R})$-valued function $(x(\cdot), u(\cdot))$ is a solution of (9.1) if, and only if, it satisfies the following integrated version of (9.1)

$$x(t) = \mathbf{T}_t x_0 + \int_0^t \mathbf{T}_{t-\tau} B(\Phi(u))(\tau)\, d\tau\,, \tag{9.2a}$$

$$u(t) = u_0 + \int_0^t k(\tau)[r - C_L x(\tau) - D(\Phi(u))(\tau)]\, d\tau\,. \tag{9.2b}$$

The next result asserts that (9.1) has a unique solution on the whole of $\mathbb{R}_+$.

Lemma 9.1.1 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $r \in \mathbb{R}$. For each $(x_0, u_0) \in X \times \mathbb{R}$, there exists a unique solution $(x(\cdot), u(\cdot))$ of (9.1) defined on $\mathbb{R}_+$.*

Proof: To recover (9.1) from (3.14), set $h \equiv 0$ and $\theta_0 = 1$ (in this case $\kappa(\cdot)$ plays the role of the gain function $k(\cdot)$). Then the result follows from Corollary 3.2.4. □

The main result of this section is contained in the following two theorems. In particular, Theorem 9.1.2 proves that if $t \mapsto k(t) > 0$ is chosen to be bounded and

† Being a Hilbert space, $X_{-1} \times \mathbb{R}$ is reflexive, and hence any absolutely continuous $(X_{-1} \times \mathbb{R})$-valued function is a.e. differentiable and can be recovered from its derivative by integration, see [2], Theorem 3.1, p. 10.

monotone decreasing to zero, then the unique solution of (9.1) is such that both $x(\cdot)$ and $(\Phi(u))(\cdot)$ converge. The essence of Theorem 9.1.4 is the assertion that if, in addition, r is such that $\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$, and $k(\cdot)$ approaches zero sufficiently slowly, then $(\Phi(u))(\cdot)$ converges to the value Φ_r, thereby ensuring asymptotic tracking of r.

Theorem 9.1.2 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A,B,C,D) \in \mathcal{L}$ and $r \in \mathbb{R}$. Let $k : \mathbb{R}_+ \to (0,\infty)$ be a bounded, monotone function with $k(t) \downarrow 0$ as $t \to \infty$. For all $(x_0,u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot),u(\cdot))$ of (9.1) satisfies*

(1) $\lim_{t\to\infty}(\Phi(u))(t)$ exists and is finite,

(2) $\lim_{t\to\infty}\|x(t) + A^{-1}B\Phi^*\| = 0\,,$ where $\Phi^* := \lim_{t\to\infty}(\Phi(u))(t)$.

Proof: Let $(x_0,u_0) \in X \times \mathbb{R}$ be arbitrary. By Lemma 9.1.1, there exists a unique solution of (9.1) on $\mathbb{R}_+$. We denote this solution by $(x(\cdot),u(\cdot))$ and introduce new variables by writing $\Phi_r = r/\mathbf{G}(0)$ and defining

$$z(t) := x(t) + A^{-1}B(\Phi(u))(t)\,, \quad v(t) := (\Phi(u))(t) - \Phi_r\,, \quad \forall\, t \in \mathbb{R}_+. \qquad (9.3)$$

By regularity, it follows that $z(t) \in \operatorname{dom}(C_L)$ for almost all $t \in \mathbb{R}_+$. For convenience we write $d_u = \Phi^\vee(u)$ (recall $\Phi^\vee$ from Definition 5.1.3) and then, by (5.4), $\dot v(t) = d_u(t)\dot u(t)$ for a.e. $t \in \mathbb{R}_+$. Since (z,v) is absolutely continuous as an $(X_{-1}\times\mathbb{R})$-valued function, we obtain by direct calculation

$$\begin{aligned} \dot v(t) &= Az(t) - k(t)d_u(t)A^{-1}B(C_Lz(t) + \mathbf{G}(0)v(t))\,, && (9.4a)\\ z(0) &= x_0 + A^{-1}B(\Phi(u))(0)\,, &&\\ \dot v(t) &= -k(t)d_u(t)(C_Lz(t) + \mathbf{G}(0)v(t))\,, && (9.4b)\\ v(0) &= (\Phi(u))(0) - \Phi_r\,. && \end{aligned}$$

We claim that there exist positive constants γ_1, γ_2 and σ_1 such that, for all t, s with $\sigma_1 \le s \le t$,

$$\int_s^t |C_Lz|\,|kd_uv| \le \gamma_1\|z(s)\|\left(\int_s^t k^2d_uv^2\right)^{1/2} + \gamma_2\int_s^t k^2d_uv^2\,. \qquad (9.5)$$

In order to prove (9.5), let us first estimate $\int_s^t |C_Lz|^2$. For notational convenience, write $w = d_u\,[C_Lz + \mathbf{G}(0)v]$. As a solution of (9.4a), $z(\cdot)$ satisfies

$$z(\tau) = \mathbf{T}_{\tau-s}z(s) - A^{-1}\int_s^\tau \mathbf{T}_{\tau-\xi}Bk(\xi)w(\xi)\,d\xi$$

for all s with $0 \leq s \leq \tau$. Invoking (3.6), (3.5) and noting that $C_L A^{-1}$ maps X boundedly into $\mathbb{R}$, there exist constants $\alpha_0, \alpha_1 > 0$ such that

$$\int_s^t |C_L z(\tau)|^2 d\tau \leq \alpha_0 \|z(s)\|^2 + \alpha_1 \int_s^t \left\| \int_s^\tau \mathbf{T}_{\tau-\xi} B k(\xi) w(\xi) d\xi \right\|^2 d\tau\,, \tag{9.6}$$

for all $0 \leq s \leq t$. By Lemma 3.1.4, part (2), interpreted in the context of the initial-value problem

$$\dot{\zeta} = A\zeta + Bkw, \quad \zeta(s) = 0,$$

we have

$$\left(\int_s^t \left\| \int_s^\tau \mathbf{T}_{\tau-\xi} B k(\xi) w(\xi) d\xi \right\|^2 d\tau \right)^{1/2} \leq \alpha_2 \left(\int_s^t |kw|^2 \right)^{1/2}$$

for some constant α_2. Therefore, by (9.6) and monotonicity of k, it follows that, for some constants $\alpha_3, \alpha_4 > 0$,

$$\begin{aligned} \left(\int_s^t |C_L z|^2 \right)^{1/2} &\leq \alpha_3 \|z(s)\| + k(s)\alpha_4 \left(\int_s^t |d_u|^2 |C_L z|^2 \right)^{1/2} \\ &\quad + \alpha_4 \mathbf{G}(0) \left(\int_s^t |k d_u v|^2 \right)^{1/2}, \quad \forall\, 0 \leq s \leq t\,. \end{aligned} \tag{9.7}$$

Fix $\sigma_1 > 0$ such that $\delta := k(\sigma_1)\alpha_4 \lambda < 1$. Then,

$$k(s)\alpha_4 \left(\int_s^t |d_u|^2 |C_L z|^2 \right)^{1/2} \leq \delta \left(\int_s^t |C_L z|^2 \right)^{1/2}, \quad \forall\, \sigma_1 \leq s \leq t\,,$$

and so, by (9.7),

$$\left(\int_s^t |C_L z|^2 \right)^{1/2} \leq \beta_1 \|z(s)\| + \beta_2 \left(\int_s^t k^2 d_u v^2 \right)^{1/2}, \quad \forall\, \sigma_1 \leq s \leq t\,, \tag{9.8}$$

with $\beta_1 = \alpha_3/(1-\delta)$ and $\beta_2 = \alpha_4 \mathbf{G}(0)\sqrt{\lambda}/(1-\delta)$. We may now deduce that, for all t, s with $\sigma_1 \leq s \leq t$,

$$\begin{aligned} \int_s^t |C_L z|\, |k d_u v| &\leq \left(\int_s^t |C_L z|^2 \right)^{1/2} \left(\int_s^t |k d_u v|^2 \right)^{1/2}, \\ &\leq \beta_1 \sqrt{\lambda} \|z(s)\| \left(\int_s^t k^2 d_u v^2 \right)^{1/2} + \beta_2 \sqrt{\lambda} \int_s^t k^2 d_u v^2\,, \end{aligned}$$

which is (9.5) with $\gamma_1 = \beta_1 \sqrt{\lambda}$ and $\gamma_2 = \beta_2 \sqrt{\lambda}$. By (9.4b), for almost all $t \geq 0$,

$$v(t)\dot{v}(t) = -k(t)\mathbf{G}(0) d_u(t) v^2(t) - k(t) d_u(t) v(t) C_L z(t)\,, \tag{9.9}$$

and hence

$$v(t)\dot{v}(t) \leq -k(t)\mathbf{G}(0)d_u(t)v^2(t) + |C_L z(t)|\,|k(t)d_u(t)v(t)|\,.$$

Integrating this inequality, and using (9.5) and monotonicity of k, yields, for all t, s with $\sigma_1 \leq s \leq t$,

$$\begin{aligned} v^2(t) \;&\leq\; v^2(s) + 2\gamma_1\sqrt{k(s)}\|z(s)\| \left(\int_s^t k d_u v^2\right)^{1/2} \\ &\quad +2\int_s^t (k\gamma_2 - \mathbf{G}(0))k d_u v^2\,. \end{aligned} \tag{9.10}$$

By positivity of $\mathbf{G}(0)$ and monotonicity of $k(\cdot)$, there exists $\sigma \geq \sigma_1$ such that, for all $\tau \geq \sigma$, $(k(\tau)\gamma_2 - \mathbf{G}(0)) \leq -\frac{1}{2}\mathbf{G}(0) < 0$. Therefore, it follows from (9.10) that

$$0 \leq v^2(\sigma) + 2\gamma_1\sqrt{k(\sigma)}\|z(\sigma)\| \left(\int_\sigma^t k d_u v^2\right)^{1/2} - \mathbf{G}(0)\int_\sigma^t k d_u v^2\,, \quad \forall\, t \geq \sigma\,,$$

and so

$$\int_\sigma^\infty k d_u v^2 < \infty\,. \tag{9.11}$$

Moreover, by (9.5) we deduce that

$$\int_\sigma^\infty |C_L z|\,|k d_u v| < \infty\,. \tag{9.12}$$

Combining (9.9), (9.11) and (9.12) shows that there exists a number $\nu \in \mathbb{R}_+$ such that

$$\lim_{t\to\infty} v^2(t) = v^2(\sigma) + 2\lim_{t\to\infty}\int_\sigma^t v\dot{v} = \nu\,,$$

whence statement (1) of the theorem. Statement (2) now follows by Lemma 3.1.4, part (1). □

Lemma 9.1.3 *Let* $\lambda > 0$. *Assume that* $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathfrak{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ *and* $r \in \mathbb{R}$ *is such that* $\Phi_r := r/\mathbf{G}(0) \in \mathrm{clos}(\mathrm{NVS}\,\Phi)$. *Let* $k : \mathbb{R}_+ \to (0,\infty)$ *be bounded and such that* $\int_0^t k =: K(t) \to \infty$ *as* $t \to \infty$. *For* $(x_0, u_0) \in X \times \mathbb{R}$, *let* $(x(\cdot), u(\cdot))$ *be the unique solution of (9.1).*
If $\lim_{t\to\infty}(\Phi(u))(t)$ *exists and is finite, then the following statements hold:*

(1) $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r\,,$

(2) $\lim_{t\to\infty}\|x(t) + A^{-1}B\Phi_r\| = 0\,,$

(3) $\lim_{t\to\infty}[r - y(t) + (\Psi_\infty x_0)(t)] = 0\,,$ *where* $y(t) = C_L x(t) + D(\Phi(u))(t)\,,$

(4) if $\Phi_r \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)$, *then* $u(\cdot)$ *is bounded.*

Proof: By hypothesis, there exists $\Phi_\infty \in \mathbb{R}$ such that $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_\infty$. The essence of the proof is to show that $\Phi_\infty = \Phi_r$. Setting

$$y_0(t) = (\Psi_\infty x_0)(t)\,, \quad y_1(t) = [\mathfrak{L}^{-1}(\mathbf{G}) \star (\Phi(u))](t)\,,$$

where $\star$ denotes convolution, we have

$$\dot{u}(t) = k(t)[r - y_0(t) - y_1(t)]\,, \quad \text{a.e. } t \in \mathbb{R}\,. \tag{9.13}$$

Since $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_\infty$ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$, it follows that

$$\lim_{t\to\infty} y_1(t) = \mathbf{G}(0)\Phi_\infty\,, \tag{9.14}$$

see [12], Theorem 6.1, part (ii), p. 96. Define a function $\tilde{y}_1 : \mathbb{R}_+ \to \mathbb{R}$ by setting

$$\tilde{y}_1(t) = r - y_1(t) = \mathbf{G}(0)\Phi_r - y_1(t)\,.$$

Seeking a contradiction, suppose that $\Phi_\infty \neq \Phi_r$. Then, either $\Phi_r > \Phi_\infty$ or $\Phi_r < \Phi_\infty$. If $\Phi_r > \Phi_\infty$, then by (9.14), there exists a number $\tau_0 \geq 0$ such that

$$\tilde{y}_1(t) \geq \frac{1}{2}\mathbf{G}(0)(\Phi_r - \Phi_\infty) > 0\,, \quad \forall\, t \geq \tau_0\,. \tag{9.15}$$

Hence, integrating (9.13) yields

$$u(t) = u(\tau) + \int_\tau^t k(s)\tilde{y}_1(s)\,ds - \int_\tau^t k(s)y_0(s)\,ds\,, \quad t \geq \tau \geq \tau_0\,. \tag{9.16}$$

Using (9.15) and estimating gives

$$\frac{1}{2}\mathbf{G}(0)(K(t) - K(\tau))(\Phi_r - \Phi_\infty) - \int_\tau^t |k(s)y_0(s)|ds \leq u(t) - u(\tau)\,, \quad \forall\, t \geq \tau\,.$$

By exponential stability, $y_0 \in L^2_\alpha(\mathbb{R}_+, \mathbb{R})$ for some $\alpha < 0$, and thus $y_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, which in turn implies that $ky_0 \in L^1(\mathbb{R}_+, \mathbb{R})$. Since $K(t) \to \infty$ as $t \to \infty$, we conclude that $u(t) \to \infty$ as $t \to \infty$. Since $ky_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, for given $\varepsilon > 0$, there exists $\tau_\varepsilon \geq \tau_0$ such that

$$\int_{\tau_\varepsilon}^\infty |k(s)y_0(s)|\,ds \leq \varepsilon\,. \tag{9.17}$$

Defining $u_\varepsilon \in C(\mathbb{R}_+, \mathbb{R})$ by

$$u_\varepsilon(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq \tau_\varepsilon\,, \\ u(\tau_\varepsilon) + \int_{\tau_\varepsilon}^t k(s)\tilde{y}_1(s)\,ds & \text{for } t > \tau_\varepsilon\,, \end{cases}$$

it follows from (9.15) that u_ε is ultimately non-decreasing, and moreover, by

(9.16) and (9.17)

$$|u(t) - u_\varepsilon(t)| \leq \varepsilon\,, \quad \forall\, t \in \mathbb{R}_+\,,$$

showing that u is approximately ultimately non-decreasing. Since $u(t) \to \infty$ as $t \to \infty$, we may invoke (C5) to conclude that

$$\Phi_r > \Phi_\infty = \lim_{t\to\infty} (\Phi(u))(t) = \sup \mathrm{NVS}\,\Phi\,,$$

which is in contradiction to the fact that $\Phi_r \in \mathrm{clos}\,(\mathrm{NVS}\,\Phi)$. If $\Phi_r < \Phi_\infty$, then a very similar argument shows that $-u$ is approximately ultimately non-decreasing and $\lim_{t\to\infty}(-u)(t) = \infty$. Invoking (C5) gives

$$\Phi_r < \Phi_\infty = \lim_{t\to\infty} (\Phi(u))(t) = \inf \mathrm{NVS}\,\Phi\,,$$

which again is in contradiction to $\Phi_r \in \mathrm{clos}\,(\mathrm{NVS}\,\Phi)$. Therefore, we may conclude that $\Phi_\infty = \Phi_r$ which is statement (1). Statement (2) follows from statement (1) and Lemma 3.1.4, part (1). For statement (3), we have

$$y(t) = C_L \mathbf{T}_t x_0 + (\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u))(t)\,. \tag{9.18}$$

By assumption $\mathfrak{L}^{-1}(\mathbf{G})$ is a finite signed Borel measure and since $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$ (by statement (1)), it follows from [12] (Theorem 6.1, part (ii), p. 96) that

$$\lim_{t\to\infty} [\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u)](t) = \mathbf{G}(0)\Phi_r = r\,.$$

Combining this with (9.18) shows that statement (3) holds. To prove statement (4), let $\Phi_r \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)$. Then, boundedness of u follows immediately from statement (1) and (C6). □

Theorem 9.1.4 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in \mathrm{clos}\,(\mathrm{NVS}\,\Phi)$. Let $k : \mathbb{R}_+ \to (0, \infty)$ be bounded, monotone and such that $k(t) \downarrow 0$ and $\int_0^t k =: K(t) \to \infty$ as $t \to \infty$. Then, we have that for all $(x_0, u_0) \in X \times \mathbb{R}$, the unique solution $(x(\cdot), u(\cdot))$ of (9.1) satisfies*

(1) $\lim_{t\to\infty}(\Phi(u))(t) = \Phi_r$,

(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\Phi_r\| = 0$,

(3) $\lim_{t\to\infty}[r - y(t) + (\mathbf{\Psi}_\infty x_0)(t)] = 0$, *where* $y(t) = C_L x(t) + D(\Phi(u))(t)$,

(4) if $\Phi_r \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)$, *then* $u(\cdot)$ *is bounded,*

(5) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and Φ_r is not a critical numerical value of Φ, then the convergence in (1) and (2) is of order $\exp(-\rho K(t))$ for some $\rho > 0$, moreover, if $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for some $\alpha < 0$, then the convergence in (3) is of order $\exp(-\rho K(t))$ for some $\rho \in (0, -\alpha)$.

Proof: Statements (1)–(4) follow immediately from Theorem 9.1.2 combined with Lemma 9.1.3. It remains only to establish statement (5). By hypothesis, $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ is not a critical numerical value of Φ and by statement (4), u is bounded. Therefore, defining $d_u = \Phi^{\vee}(u)$ (recall $\Phi^{\vee}$ from Definition 5.1.3), there exists $\sigma_1 > 0$ and $d > 0$ such that

$$d_u(t) \in [d, \lambda], \quad \text{a.e. } t \geq \sigma_1 . \tag{9.19}$$

Define $\rho := \frac{1}{2}\mathbf{G}(0)d > 0$ and introduce exponentially weighted variables given by

$$\begin{aligned} z_e(t) &:= \exp(\rho K(t))[x(t) + A^{-1}B(\Phi(u))(t)] , & (9.20\text{a}) \\ v_e(t) &:= \exp(\rho K(t))[(\Phi(u))(t) - \Phi_r] , & (9.20\text{b}) \end{aligned}$$

for all $t \in \mathbb{R}_+$. Since (z_e, v_e) is absolutely continuous as an $(X_{-1} \times \mathbb{R})$-valued function and using the fact that $(\Phi(u))'(t) = d_u(t)\dot{u}(t)$ for a.e. $t \in \mathbb{R}_+$ (by (5.4)), we obtain by direct calculation

$$\begin{aligned} \dot{z}_e(t) &= (A + \rho k(t)I)z(t) - k(t)d_u(t)A^{-1}B(C_L z(t) + \mathbf{G}(0)v(t)) , & (9.21\text{a}) \\ z_e(0) &= x_0 + A^{-1}B(\Phi(u))(0) , & \\ \dot{v}_e(t) &= -k(t)d_u(t)(C_L z(t) + \mathbf{G}(0)v(t)) + \rho k(t)v(t) , & (9.21\text{b}) \\ v_e(0) &= (\Phi(u))(0) - \Phi_r . & \end{aligned}$$

For each (t, s) with $0 \leq s \leq t$, define

$$\mathbf{U}(t,s) := \exp(\rho[K(t) - K(s)])\mathbf{T}_{t-s} . \tag{9.22}$$

We state the following lemma which was proved in [24] (see Lemma 3.10 in [24]).

Lemma 9.1.5 *Let $s \in \mathbb{R}_+$, $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and, on $[s, \infty)$, define a function p by*

$$p(t) := \int_s^t \mathbf{U}(t,\xi)Bu(\xi)d\xi.$$

Then, for all $t \in [s, \infty)$, $p(t) \in X$ and, as an X_{-1}-valued function, p is absolutely continuous with

$$\dot{p}(t) = (A + \rho k(t)I)p(t) + Bu(t) \quad \text{a.e. } t \geq s .$$

Returning to the proof of the theorem, for notational convenience write

$$w_e := d_u\left[C_L z_e + \mathbf{G}(0)v_e\right].$$

Let $s \in \mathbb{R}_+$ and, on $[s,\infty)$, define $f := f_1 - A^{-1}f_2$ with

$$f_1(t) := \mathbf{U}(t,s)z_e(s), \qquad f_2(t) := \int_s^t \mathbf{U}(t,\xi)Bk(\xi)w_e(\xi)d\xi\,.$$

Clearly, $f_1(t) \in X$ for all $t \in [s,\infty)$ and, as an X_{-1}-valued function, f_1 is absolutely continuous with

$$\dot{f}_1(t) = (A + \rho k(t)I)f_1(t) \quad \text{a.e. } t \in [s,\infty)\,.$$

By Lemma 9.1.5, it now follows that $f(t) \in X$ for all $t \in [s,\infty)$ and, as an X_{-1}-valued function, f is absolutely continuous with

$$\begin{aligned}\dot{f}(t) &= (A + \rho k(t)I)f_1(t) - A^{-1}\left((A + \rho k(t)I)f_2(t) + Bw_e(t)\right)\\ &= (A + \rho k(t)I)f(t) - A^{-1}Bw_e(t) \quad \text{a.e. } t \in [s,\infty)\,.\end{aligned}$$

In view of (9.21a) (together with uniqueness of solutions), we may now conclude that

$$z_e(t) = \mathbf{U}(t,s)z_e(s) - A^{-1}\int_s^t \mathbf{U}(t,\xi)Bk(\xi)w_e(\xi)\,d\xi\,, \quad \forall\, t,s \text{ with } 0 \le s \le t\,. \tag{9.23}$$

By exponential stability of the semigroup $\mathbf{T}$, there exist constants N, $\theta > 0$ such that $\|\mathbf{T}_t\| \le N\exp(-\theta t)$ for all $t \in \mathbb{R}_+$. Let $\varepsilon \in (0,\theta)$ be sufficiently small such that $(A + \varepsilon I, B, C, D) \in \mathfrak{L}$ (recall Proposition 3.1.3). Fix $\sigma_2 > \sigma_1$ such that

$$k(\sigma_2) < \min\{\varepsilon/\rho, \theta/(\rho N)\}. \tag{9.24}$$

Again, we digress to state a technicality which was proved in [24] (see Lemma 3.11 in [24]).

Lemma 9.1.6 *There exists constant $\gamma > 0$ such that, for all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$,*

$$\left(\int_s^t \left\|\int_s^\tau \mathbf{U}(\tau,\xi)Bu(\xi)d\xi\right\|^2 d\tau\right)^{1/2} \le \gamma\left(\int_s^t u^2(\xi)d\xi\right)^{1/2} \quad \forall\, s,t \ge \sigma_2 \textit{ with } s \le t.$$

Once more, we return to the proof of the theorem. By monotonicity of k, $K(t)$ –

$K(s) \leq k(s)(t-s)$ for all t, s with $0 \leq s \leq t$. Since $k(\sigma_2) \leq \varepsilon/\rho$, it follows that

$$\exp(\rho[K(t) - K(s)]) \leq \exp(\varepsilon[t-s]) \quad \text{for all } t, s \text{ with } \sigma_2 \leq s \leq t. \tag{9.25}$$

Observe that, for all t, s with $\sigma_2 \leq s \leq t$,

$$\begin{aligned} |C_L \mathbf{U}(t,s) z_e(s)| &= |C_L \mathbf{T}_{t-s} z_e(s)| \exp(\rho[K(t) - K(s)]) \\ &\leq |C_L \mathbf{T}_{t-s} \exp(\varepsilon[t-s]) z_e(s)| \,. \end{aligned}$$

Invoking (3.6), (3.5) (in the context of the regular system $(A + \varepsilon I, B, C, D)$), (9.23), Lemma 9.1.6, and recalling that $C_L A^{-1}$ maps X boundedly into $\mathbb{R}$, there exist constants $\alpha_2, \alpha_3 > 0$ such that

$$\begin{aligned} \left(\int_s^t |C_L z_e|^2 \right)^{1/2} &\leq \alpha_2 \|z_e(s)\| + k(s) \alpha_3 \left(\int_s^t |d_u|^2 |C_L z_e|^2 \right)^{1/2} \\ &\quad + \alpha_3 \mathbf{G}(0) \left(\int_s^t |k d_u v_e|^2 \right)^{1/2} \end{aligned} \tag{9.26}$$

for all t, s with $\sigma_2 \leq s \leq t$.

Inequality (9.26) is the exponentially weighted version of (9.7). Following the argument in the proof of Theorem 9.1.2, (9.26) may be used to derive an exponentially weighted version of (9.5), i.e. there exist positive constants $\gamma_1, \gamma_2 > 0$ and $\sigma_3 \geq \sigma_2$ such that

$$\int_s^t |C_L z_e|\, |k d_u v_e| \leq \gamma_1 \|z_e(s)\| \left(\int_s^t k^2 d_u v_e^2 \right)^{1/2} + \gamma_2 \int_s^t k^2 d_u v_e^2 \tag{9.27}$$

for all t, s with $\sigma_3 \leq s \leq t$.

By (9.21b), for almost all $t \geq 0$,

$$v_e(t) \dot{v}_e(t) = -k(t) \mathbf{G}(0) d_u(t) v_e^2(t) + \rho k(t) v_e^2(t) - k(t) d_u(t) v_e(t) C_L z_e(t) \,. \tag{9.28}$$

By (9.19), $\mathbf{G}(0) d_u(t) - \rho \geq \mathbf{G}(0) d - \rho = \rho > 0$ for all $t \geq \sigma_3$. Hence, we have

$$v_e(t) \dot{v}_e(t) \leq -\frac{1}{2} \rho k(t) v_e^2(t) + |C_L z_e(t)|\, |k(t) d_u(t) v_e(t)| \quad \text{for a.e. } t \geq \sigma_3 \,.$$

Integrating this inequality, and using (9.27) and monotonicity of k, yields, for all t, s with $t \geq s \geq \sigma_3$,

$$v_e^2(t) \leq v_e^2(s) + 2\gamma_1 \sqrt{\lambda k(s)} \|z_e(s)\| \left(\int_s^t k v_e^2 \right)^{1/2} - \int_s^t (\rho - 2k\gamma_2 \lambda) k v_e^2 \,. \tag{9.29}$$

Fix $\sigma \geq \sigma_3$ such that $\rho - 2k(t)\gamma_2 \lambda > \frac{1}{2}\rho$ for all $t \geq \sigma$. From (9.29) and (9.27), we deduce

$$\int_{\sigma}^{\infty} kv_e^2 < \infty\,.$$

Hence, by (9.29), $v_e(\cdot) = \exp(\rho K(\cdot))[(\Phi(u))(\cdot) - \Phi_r]$ is bounded and so the convergence in (1) is of order $\exp(-\rho K(t))$. We proceed to prove that the convergence in (2) is of the same order. Define $x_r := -A^{-1}B\Phi_r$, and introduce a new variable given by

$$x_e(t) = \exp(\rho K(t))[x(t) - x_r]\,, \quad \forall\, t \geq 0\,.$$

It suffices to show that $x_e(\cdot)$ is bounded. By (9.1a) and (9.20), we have

$$\dot{x}_e = (A + \rho k I)x_e + Bv\,, \quad x_e(0) = x_0 - x_r\,.$$

and so, for all $t \geq \sigma$

$$x_e(t) = \mathbf{T}_{t-\sigma}x_e(\sigma) + \int_{\sigma}^{t} \mathbf{T}_{t-\xi}Bv_e(\xi)d\xi + \int_{\sigma}^{t} \mathbf{T}_{t-\xi}\rho k(\xi)x_e(\xi)d\xi\,.$$

Therefore, by boundedness of v_e together with Lemma 3.1.4, part (4), and exponential stability of $\mathbf{T}$, there exists a constant $\beta > 0$ such that

$$\sup_{s\in[\sigma,t]} \|x_e(s)\| \leq \beta + \rho N\theta^{-1}k(\sigma) \sup_{s\in[\sigma,t]} \|x_e(s)\|\,, \quad \forall\, t \geq \sigma\,.$$

Since $\sigma \geq \sigma_2$, we have, by (9.24), $\rho N\theta^{-1}k(\sigma) < 1$, and hence we may conclude boundedness of x_e. Therefore the convergence in part (2) is of order $\exp(-\rho K(t))$

Suppose that $\mu := \mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for some $\alpha < 0$. It remains only to prove that the convergence in (3) is also of order $\exp(-\rho K(t))$ for some $\rho \in (0, -\alpha)$. Recalling that the unit-step function is denoted by U, we have for all $t \in \mathbb{R}_+$

$$|r - y(t) + (\Psi_{\infty}x_0)(t)| \leq |[\mu \star (\Phi(u) - \Phi_r U)(t)]| + |\Phi_r[(\mu \star U)(t) - \mathbf{G}(0)]|\,. \quad (9.30)$$

For convenience we set $g(t) = \exp(\rho K(t))$ for all $t \geq 0$. We have already shown that the function $t \mapsto g(t)|(\Phi(u))(t) - \Phi_r|$ remains bounded as $t \to \infty$. If we extend g to a function defined on $\mathbb{R}$ by setting $g(t) = 1$ for all $t < 0$, then it is easy to show that g is a submultiplicative weight function in the sense of [12], p. 118. Moreover, since $\mu \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$, the measure $\mu_g : E \mapsto \int_E g(t)\, d\mu(t)$ belongs to $\mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$. Hence, by [12] (Theorem 3.5, part (i), p. 119), we may conclude that the function $t \mapsto g(t)[\mu \star (\Phi(u) - \Phi_r U)](t)$ is bounded on $\mathbb{R}_+$.

Since $\mu_g \in \mathcal{M}_{\mathrm{f}}(\mathbb{R}_+)$ (a space of finite measures), $\int_0^{\infty} g(t)\, d|\mu|(t) < \infty$. Hence

$$|g(t)[(\mu \star U)(t) - \mathbf{G}(0)]| = g(t)\left|\int_t^{\infty} d\mu(\tau)\right| \leq \int_0^{\infty} g(\tau)\, d|\mu|(\tau) < \infty\,,$$

showing that the function $t \mapsto g(t)[(\mu \star U)(t) - \mathbf{G}(0)]$ is bounded on $\mathbb{R}_+$. Consequently, appealing to (9.30), we deduce that the function

$$\mathbb{R}_+ \to \mathbb{R}, \quad t \mapsto \exp(\rho K(t))|r - y(t) + (\Psi_\infty x_0)(t)|$$

is bounded. □

9.2 Continuous-time integral control with adaptive gain in the presence of input nonlinearities in $\mathcal{N}_c(\lambda)$

Whilst Theorem 9.1.4 identifies conditions under which the tracking objective is achieved through the use of a monotone gain function, the resulting control strategy is somewhat unsatisfactory insofar as the gain function is selected *a priori:* no use is made of the output information from the plant to update the gain. We now consider the possibility of exploiting this output information to generate, by feedback, an appropriate gain function. In particular, let the gain $k(\cdot)$ be generated by the law:

$$k(t) = \frac{1}{l(t)}, \quad \dot{l}(t) = |r - y(t)|, \quad l(0) = l_0 > 0. \tag{9.31}$$

which yields the feedback system

$$\dot{x}(t) = Ax(t) + B(\Phi(u))(t), \quad x(0) = x_0 \in X, \tag{9.32a}$$
$$\dot{u}(t) = (1/l(t))[r - C_L x(t) - D(\Phi(u))(t)], \quad u(0) = u_0 \in \mathbb{R}, \tag{9.32b}$$
$$\dot{l}(t) = |r - C_L x(t) - D(\Phi(u))(t)|, \quad l(0) = l_0 \in (0, \infty). \tag{9.32c}$$

The concept of a solution to this feedback system is the obvious modification of the solution concept defined in the previous section.

Lemma 9.2.1 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$ and $r \in \mathbb{R}$. For each $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, the initial-value problem given by (9.32) has a unique solution defined on $\mathbb{R}_+$.*

Proof: First note that, by setting $k(t) = 1/l(t)$, the adaptive feedback system (9.32) (with $(A, B, C, D) \in \mathcal{L}$) can be written in the following form

$$\dot{x}(t) = Ax(t) + B(\Phi(u))(t)\,, \quad x(0) = x_0 \in X\,, \tag{9.33a}$$

$$\dot{u}(t) = k(t)[r - C_L x(t) - D(\Phi(u))(t)]\,, \quad u(0) = u_0 \in \mathbb{R}\,, \tag{9.33b}$$

$$\dot{k}(t) = -k^2(t)|r - C_L x(t) - D(\Phi(u))(t)|\,, \quad k(0) = k_0 \in (0,\infty)\,. \tag{9.33c}$$

We can recover (9.33) from (3.14), by considering the special case $\kappa(t) \equiv 1$ and $h(\theta) = -\theta^2$, which gives the adaptive feedback equations (9.33) (with $k(\cdot) = \theta(\cdot)$). Thus any application of Corollary 3.2.4 completes the proof. □

We now arrive at the main adaptive control result.

Theorem 9.2.2 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_c(\lambda)$, $(A, B, C, D) \in \mathcal{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$. For all $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0,\infty)$, the unique solution, (x, u, l), of the initial-value problem given by (9.32) is such that statements (1)–(4) of Theorem 9.1.4 hold. Moreover, if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$, Φ_r is not a critical numerical value of Φ and $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f^{\alpha}(\mathbb{R}_+)$ for some $\alpha < 0$, then the monotone gain $k(t) = 1/l(t)$ converges to a positive value and the input $u(t)$ converges to a finite value, as $t \to \infty$.*

Proof: Set $k(t) = 1/l(t)$. Since $l(\cdot)$ is monotone increasing, either $l(t) \to \infty$ as $t \to \infty$ (Case 1), or $l(t) \to l^* \in (0,\infty)$ as $t \to \infty$ (Case 2). We consider these two cases separately.

CASE 1. In this case, $k(t) \downarrow 0$ as $t \to \infty$ and the hypotheses of Theorem 9.1.2 are satisfied. Therefore, $(\Phi(u))(\cdot)$ converges. It follows that $\lim_{t\to\infty}(\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u))(t)$ converges (and so, in particular, is a bounded function). Moreover, by exponential stability, $\Psi_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, and it follows from

$$\dot{l}(t) = |r - y(t)| \le |r - (\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u))(t)| + |(\Psi_\infty x_0)(t)|\,,$$

via integration that

$$k(t) = \frac{1}{l(t)} \ge \frac{1}{\alpha + \beta t} \quad \forall\, t \ge 0\,, \tag{9.34}$$

where

$$\alpha := l_0 + \int_0^\infty |\Psi_\infty x_0(\tau)|\, d\tau\,, \quad \beta \ge \sup_{t\ge 0} |r - (\mathfrak{L}^{-1}(\mathbf{G}) \star \Phi(u))(t)|\,.$$

Therefore, statements (1)–(4) of Theorem 9.1.4 hold.

CASE 2. In this case, $k(t) \to k^* := 1/l^* > 0$ as $t \to \infty$. By boundedness of $l(\cdot)$ and (9.31), we may conclude that $e(\cdot) := r - C_L x(\cdot) - D(\Phi(u))(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R})$ and so (by (9.1b)) $u(t)$ converges to a finite limit as $t \to \infty$. Consequently, $(\Phi(u))(t)$

converges to a finite limit as $t \to \infty$, and hence, by Lemma 9.1.3, statements (1)–(4) of Theorem 9.1.4 hold.

Finally, assume that $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ is not a critical numerical value of Φ and that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for some $\alpha < 0$. We will show that the monotone gain k converges to a positive value. Seeking a contradiction, suppose that the monotone function l is unbounded (equivalently, $k(t) \downarrow 0$ as $t \to \infty$). Then the hypotheses of Theorem 9.1.2 are satisfied and so (9.34) holds. By Theorem 9.1.4, $(\Phi(u))(\cdot)$ converges to Φ_r and $y(\cdot) - (\Psi_\infty x_0)(\cdot)$ converges to r; moreover, the convergence is of order $\exp(-\rho K(t))$ for some $\rho > 0$, that is, there exists constant $L > 0$ such that

$$|r - y(t) + (\Psi_\infty x_0)(t)| \leq L \exp(-\rho K(t)), \quad \forall\, t \in \mathbb{R}_+ . \tag{9.35}$$

Choose $\gamma \geq \beta$ such that $\rho/\gamma < 1$. By (9.34), $k(t) = 1/l(t) \geq (\alpha + \gamma t)^{-1}$ for all $t \in \mathbb{R}_+$. Therefore,

$$K(t) = \int_0^t k \;\geq\; \ln[((\alpha + \gamma t)/\alpha)^{1/\gamma}], \quad \forall\, t \geq 0 .$$

Consequently for almost all $t \geq 0$,

$$\dot{l}(t) = |r - y(t)| \leq L \exp(-\rho K(t)) + |(\Psi_\infty x_0)(t)| \leq M(\alpha + \gamma t)^{-\eta} + |(\Psi_\infty x_0)(t)|$$

where $\eta = \rho/\gamma \in (0,1)$ and $M = L\alpha^{\eta}$. Since, by exponential stability, $\Psi_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, integration gives

$$l(t) \leq N(\alpha + \gamma t)^{1-\eta}, \quad \forall\, t \geq 0 ,$$

for some suitable constant $N > 0$. It follows that

$$-K(t) = -\int_0^t k \leq -(N\gamma\eta)^{-1}\left[(\alpha + \gamma t)^{\eta} - \alpha^{\eta}\right], \quad \forall\, t \geq 0 .$$

Therefore, $\exp(-\rho K(\cdot))$ is of class $L^1(\mathbb{R}_+, \mathbb{R})$ and, by (9.35), it follows that $|r - y(\cdot) + (\Psi_\infty x_0)(\cdot)|$ is also of class $L^1(\mathbb{R}_+, \mathbb{R})$. Since $\Psi_\infty x_0 \in L^1(\mathbb{R}_+, \mathbb{R})$, we have $|r - y(\cdot)| \in L^1(\mathbb{R}_+, \mathbb{R})$. This contradicts the supposition of unboundedness of $l(\cdot)$. Therefore, $l(\cdot)$ is bounded. Since $l(\cdot)$ is bounded, we may conclude from (9.31), that $\dot{u} = r - y \in L^1(\mathbb{R}_+, \mathbb{R})$ and therefore $u(t)$ converges to a finite number as $t \to \infty$. $\square$

9.3 Discrete-time and sampled-data integral control with adaptive gain in the presence of input nonlinearities in $\mathcal{N}_d(\lambda)$ and $\mathcal{N}_{sd}(\lambda)$

In this section we first present the discrete-time counterpart of the continuous-time result contained in Section 9.2. We do not include a proof since it would differ little from the proof of Theorem 2.7 in [23] (see Theorem 2.7 and the preceding results in [23]). We then use the discrete-time result to prove a sampled-data result in much the same way as was done in Chapter 8.

Discrete-time control

As in Chapter 7 we consider the nonlinear system (7.2). Denoting the reference value by r, the control law

$$\begin{aligned} u(n+1) &= u(n) + (1/l(n))(r - y(n)), \quad u(0) = u_0 \in \mathbb{R}, \\ l(n+1) &= l(n) + |r - y(n)|, \quad l(0) = l_0 > 0, \end{aligned}$$

yields the feedback system

$$x(n+1) = Ax(n) + B(\Phi(u))(n), \quad x(0) = x_0 \in X, \tag{9.36a}$$

$$u(n+1) = u(n) + (1/l(n))[r - Cx(n) - D(\Phi(u))(n)], \quad u(0) = u_0 \in \mathbb{R}, \tag{9.36b}$$

$$l(n+1) = l(n) + |r - Cx(n) - D(\Phi(u))(n)|, \quad l(0) = l_0 \in (0, \infty). \tag{9.36c}$$

Theorem 9.3.1 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_d(\lambda)$, A is power-stable,* $\mathbf{G}(1) > 0$ *and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(1) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$. For all $(x_0, u_0, l_0) \in X \times \mathbb{R} \times (0, \infty)$, the unique solution (x, u, l) of (9.36) satisfies*

(1) $\lim_{n\to\infty} (\Phi(u))(n) = \Phi_r$,

(2) $\lim_{n\to\infty} x(n) = (I - A)^{-1} B \Phi_r$,

(3) $\lim_{n\to\infty} y(n) = r$, *where* $y(n) = Cx(n) + D(\Phi(u))(n)$,

(4) if $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$, *then u is bounded,*

(5) if $\Phi_r \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$ *and Φ_r is not a critical numerical value of Φ, then the monotone gain $k(n) = 1/l(n)$ converges to a positive value and the input $u(n)$ converges to a finite value, as $n \to \infty$.*

Sampled-data control

Let $\Phi \in \mathcal{N}_{sd}(\lambda)$, $(A,B,C,D) \in \mathcal{L}$ and let $\tilde{\Phi}$ denote the extension of Φ to $NPC_{\mathrm{pm}}(\mathbb{R}_+,\mathbb{R})$ given by (4.31). As in Chapter 8 we distinguish two cases: bounded and unbounded observation.

Bounded observation

Assume that $C = C_L \in L(X,\mathbb{R})$. Consider the nonlinear system (8.11) controlled by the sampled-data integrator

$$\begin{aligned} u(t) &= u^d(n)\,, \quad \text{for } t \in [n\tau,(n+1)\tau),\, n \in \mathbb{Z}_+\,, && (9.37\text{a})\\ y^d(n) &= y(n\tau)\,, \quad n \in \mathbb{Z}_+\,, && (9.37\text{b})\\ u^d(n+1) &= u^d(n) + (1/l^d(n))(r - y^d(n))\,, \quad u^d(0) = u_0 \in \mathbb{R},\, n \in \mathbb{Z}_+\,, && (9.37\text{c})\\ l^d(n+1) &= l^d(n) + |r - y^d(n)|\,, \quad l^d(0) = l_0^d > 0\,. && (9.37\text{d}) \end{aligned}$$

The following results follow from Theorem 9.3.1 exactly as Theorem 8.1.2 follows from Theorem 7.1.4.

Theorem 9.3.2 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_{sd}(\lambda)$, $(A,B,C,D) \in \mathcal{L}$, C is bounded and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\mathrm{NVS}\,\Phi)$. For all $(x_0,u_0,l_0^d) \in X\times\mathbb{R}\times(0,\infty)$, the unique solution $(x(\cdot),u(\cdot),l^d(\cdot))$ of the closed-loop system given by (8.11) and (9.37) satisfies*

(1) $\lim_{t\to\infty}(\tilde{\Phi}(u))(t) = \Phi_r$ *,*

(2) $\lim_{t\to\infty}\|x(t) + A^{-1}B\Phi_r\| = 0$ *,*

(3) $\lim_{t\to\infty} y(t) = r$ *,*

(4) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$, then u is bounded,

(5) if $\Phi_r \in \operatorname{int}(\mathrm{NVS}\,\Phi)$ and Φ_r is not a critical numerical value of Φ, then the monotone gain $k^d(n) = 1/l^d(n)$ converges to a positive value as $n \to \infty$ and the input $u(t)$ converges to a finite value as $t \to \infty$.

Unbounded observation

Consider the following sampled-data low-gain controller for (8.11)

$$
\begin{aligned}
u(t) &= u^d(n)\,, \quad \text{for } t \in [n\tau, (n+1)\tau),\, n \in \mathbb{Z}_+\,, && \text{(9.38a)}\\
y^d(n) &= \int_0^\tau w(t) y(n\tau + t)\, dt\,, \quad n \in \mathbb{Z}_+\,, && \text{(9.38b)}\\
u^d(n+1) &= u^d(n) + (1/l^d(n))(r - y^d(n))\,, \quad u(0) = u_0 \in \mathbb{R},\, n \in \mathbb{Z}_+\,, && \text{(9.38c)}\\
l^d(n+1) &= l^d(n) + |r - y^d(n)|\,, \quad l^d(0) = l_0^d > 0\,, && \text{(9.38d)}
\end{aligned}
$$

where $w \in L^2([0,\tau], \mathbb{R})$ satisfies (8.17).

The following result follows from Theorem 9.3.1 exactly as Theorem 8.1.5 follows from Theorem 7.1.4.

Theorem 9.3.3 *Let $\lambda > 0$. Assume that $\Phi \in \mathcal{N}_{sd}(\lambda)$, $(A, B, C, D) \in \mathfrak{L}$, $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_f(\mathbb{R}_+)$ and $r \in \mathbb{R}$ is such that $\Phi_r := r/\mathbf{G}(0) \in \operatorname{clos}(\operatorname{NVS}\Phi)$. For all $(x_0, u_0, l_0^d) \in X \times \mathbb{R} \times (0, \infty)$, the unique solution $(x(\cdot), u(\cdot), l^d(\cdot))$ of the closed-loop system given by (8.11) and (9.38) satisfies*

(1) $\lim_{t\to\infty} (\tilde{\Phi}(u))(t) = \Phi_r$,

(2) $\lim_{t\to\infty} \|x(t) + A^{-1}B\Phi_r\| = 0$,

(3) $\lim_{t\to\infty} [r - y(t) + C_L \mathbf{T}_t x_0] = 0$,

(4) if $\Phi_r \in \operatorname{int}(\operatorname{NVS}\Phi)$, then u is bounded,

(5) if $\Phi_r \in \operatorname{int}(\operatorname{NVS}\Phi)$ and Φ_r is not a critical numerical value of Φ, then the monotone gain $k^d(n) = 1/l^d(n)$ converges to a positive value as $n \to \infty$ and the input $u(t)$ converges to a finite value as $t \to \infty$.

9.4 Example: controlled diffusion process with output delay

Consider a diffusion process (with diffusion coefficient $\kappa > 0$ and with Dirichlet boundary conditions), on the one-dimensional spatial domain $[0, 1]$, with scalar nonlinear pointwise control action (applied at point $x_1 \in (0, 1)$, via an operator $\Phi : C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, as defined below) and delayed (delay $T \geq 0$) pointwise scalar observation (output at point $x_2 \in (x_1, 1)$).

We formally write this single-input, single-output system as

$$\begin{aligned} z_t(t,x) &= \kappa z_{xx}(t,x) + \delta(x-x_1)(\Phi(u))(t)\,, \\ y(t) &= z(t-T,x_2)\,, \end{aligned}$$

with boundary conditions

$$z(t,0) = 0 = z(t,1)\,, \quad \forall\, t > 0\,.$$

For simplicity, we assume zero initial conditions

$$z(t,x) = 0\,, \quad \forall\,(t,x) \in [-T,0] \times [0,1]\,.$$

With input $(\Phi(u))(\cdot)$ and output $y(\cdot)$, this example qualifies as a regular linear system with transfer function given by

$$\mathbf{G}(s) = \frac{e^{-sT}\sinh\left(x_1\sqrt{s/\kappa}\right)\sinh\left((1-x_2)\sqrt{s/\kappa}\right)}{\kappa\sqrt{s/\kappa}\sinh\sqrt{s/\kappa}}\,.$$

It is not difficult to show that $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for any $\alpha > -\kappa\pi^2$ (see Appendix 6 for details).

For purposes of illustration, we adopt the following values

$$\kappa = 0.1, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad T = 1\,.$$

Let $\Phi = \mathcal{E}_{1.1,0}$ be the elastic-plastic operator as defined in Section 4.3. Then $\Phi \in \mathcal{N}_c(\lambda)$ where $\lambda = 2$ and $\mathrm{NVS}\,\Phi = [-1.1, 1.1]$. For reference value $r = 1$

$$\Phi_r = \frac{r}{\mathbf{G}(0)} = \frac{r\kappa}{x_1(1-x_2)} = 0.9 \in \mathrm{int}\,(\mathrm{NVS}\,\Phi)\,.$$

We consider both continuous-time integral control with adaptive gain (a) and sampled-data integral control with adaptive gain (b).
(a) By Theorem 6.1.2, since $\Phi \in \mathcal{N}_c(2)$, the adaptive integral control,

$$u(t) = \int_0^t k(t)[r - y(t)]\,dt\,, \qquad k(t) = \frac{1}{l(t)}\,,$$

where the evolution of $l(t)$ is given by the adaptation law

$$\dot{l}(t) = |r - y(t)|\,, \quad l(0) = l_0 > 0\,,$$

guarantees asymptotic tracking.

In each of the following three cases

$$\text{(i) } l_0 = 1 \text{ (solid)}, \quad \text{(ii) } l_0 = 0.5 \text{ (dotdash)}, \quad \text{(iii) } l_0 = 0.2 \text{ (dotted)},$$

Figure 39 depicts the output behaviour of the system under integral control, Figure 40 depicts the corresponding control input, Figure 41 shows the input of the elastic-plastic operator and Figure 42 shows the time-varying gain. Since Φ_r is not a critical value of $\mathcal{E}_{1.1,0}$, the gain $k(t)$ converges to a positive value and $u(t)$ converges to a finite value, as $t \to \infty$.

(b) We have unbounded observation in the above diffusion equation and therefore we can apply Theorem 9.3.3. We adopt the generalized sampling operation given by (8.18) with $w(\cdot) \equiv 1/\tau$:

$$y(n) = \frac{1}{\tau}\int_0^\tau y(n\tau + t)\, dt\,.$$

Therefore, by Theorem 9.3.3, for each $(u_0, l_0^d) \in \mathbb{R}\times(0,\infty)$, the adaptive sampled-data control (with sampling at times $n\tau$, $\tau > 0$, and hold on intervals $[n\tau, (n+1)\tau))$ given by

$$\begin{aligned}
u(t) &= u^d(n)\,, \quad \text{for } t \in [n\tau, (n+1)\tau),\, n \in \mathbb{Z}_+\,,\\
y^d(n) &= \frac{1}{\tau}\int_0^\tau y(n\tau + t)\, dt\,, \quad n \in \mathbb{Z}_+\,,\\
u^d(n+1) &= u^d(n) + (1/l^d(n))(r - y^d(n))\,, \quad u(0) = u_0\,,\, n \in \mathbb{Z}_+\,,\\
l^d(n+1) &= l^d(n) + |r - y^d(n)|\,, \quad l^d(0) = l_0^d\,,
\end{aligned}$$

guarantees asymptotic tracking.

In both of the following two cases

$$\text{(i) } l_0 = 1 \text{ (solid)}, \quad \text{(ii) } l_0 = 0.2 \text{ (dotted)},$$

Figure 43 depicts the output behaviour of the system under sampled-data control, Figure 44 depicts the corresponding control input, Figure 45 shows the input of the elastic-plastic operator and Figure 46 shows the time-varying gain sequence. Since Φ_r is not a critical value of $\mathcal{E}_{1.1,0}$, the gain $k(n)$ converges to a positive value as $n \to \infty$ and $u(t)$ converges to a finite value, as $t \to \infty$.

Figures 39–46 were generated using SIMULINK Simulation Software within MATLAB wherein a truncated eigenfunction expansion, of order 10, was adopted to model the diffusion process.

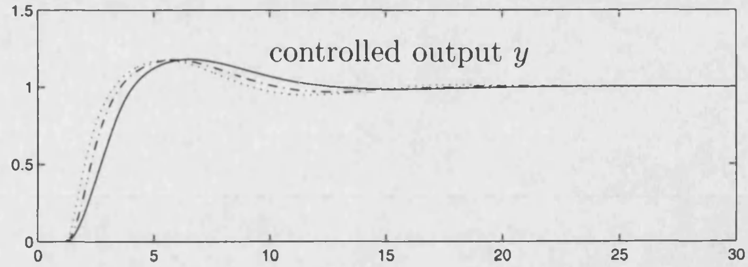

Figure 39: Controlled output

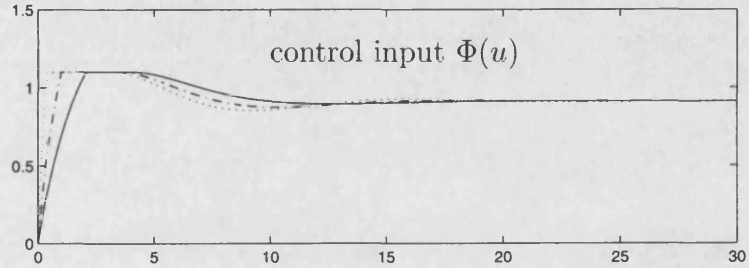

Figure 40: Control input

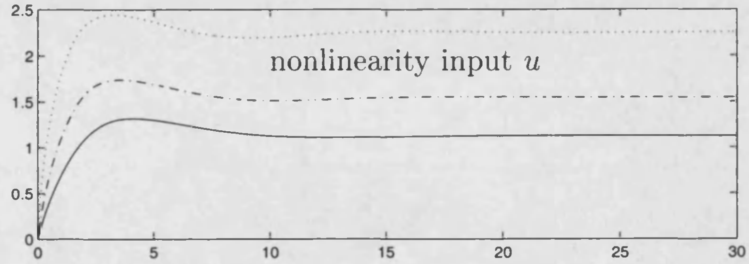

Figure 41: Input of elastic-plastic operator

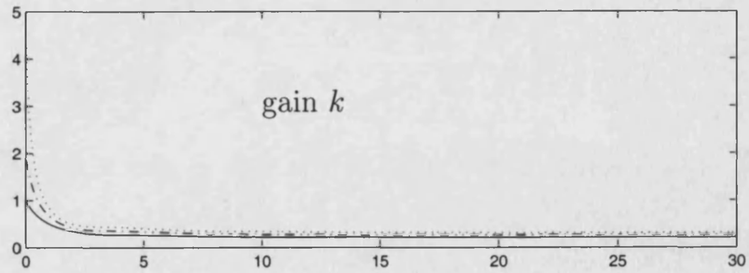

Figure 42: Time-varying gain

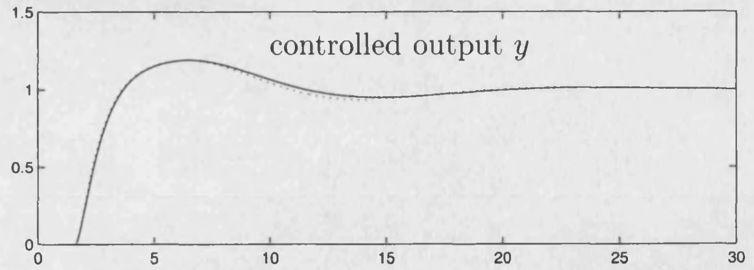

Figure 43: Controlled output

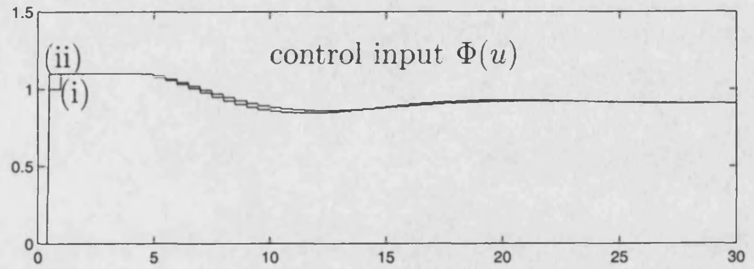

Figure 44: Control input

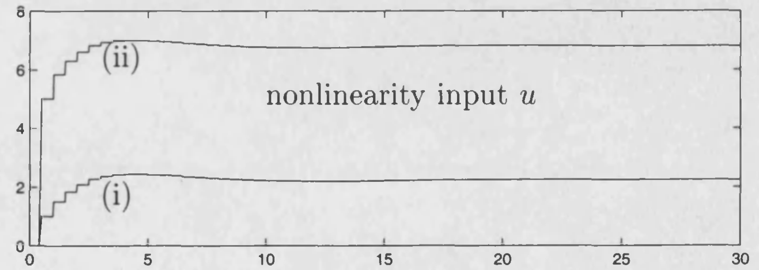

Figure 45: Input of elastic-plastic operator

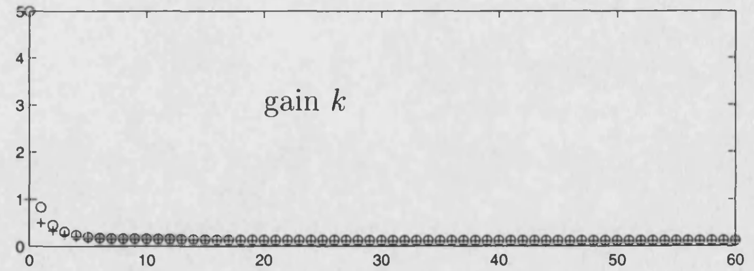

Figure 46: Time-varying gain sequence

9.5 Notes and references

The results in this chapter are all new. The proofs in Sections 9.1 and 9.2 are similar to the proofs of the corresponding results for static nonlinearities in [24]. Theorem 9.3.1 can be proved in a similar way to Theorem 2.7 in [23].

Chapter 10

Appendices

Appendix 1: $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ is not a linear space

Define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0\,, & t = 0\,, \\ \frac{1}{2n+2} + \frac{\frac{1}{2n} - \frac{1}{2n+2}}{\frac{1}{2n} - \frac{1}{2n+1}} \left(t - \frac{1}{2n+1}\right)\,, & t \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)\,, \quad n \in \mathbb{N}\,, \\ \frac{1}{2n}\,, & t \in \left[\frac{1}{2n}, \frac{1}{2n-1}\right]\,, \quad n \in \mathbb{N}\,, \\ \frac{1}{2}\,, & t > 1\,. \end{cases}$$

Clearly $f \in C(\mathbb{R}_+, \mathbb{R})$ and f is non-decreasing. Define the function $g : \mathbb{R}_+ \to \mathbb{R}$, $g : t \mapsto -t$. Since f and g are both monotone, $f, g \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. We see that $f+g$ is not piecewise monotone since for each $n \in \mathbb{N}$ it has a strict local maximum at $1/(2n)$.

Appendix 2: Proof that V, given by (3.18), is weakly locally Lipschitz

For convenience we write $g(t) = r - (\Psi_\infty x_0)(t)$. Let $\alpha \geq 0$ and $w \in C([0, \alpha], \mathbb{R}^2)$ and write $w = (u_0, \theta_0)^T$. Let $\delta_1, \delta_2, \gamma_1, \gamma_2 > 0$ be such that (3.16) and (3.17) hold for all $\varepsilon \in [0, \delta_2]$ and all $u, v \in C(u_0; \delta_1, \delta_2)$. Fix $\varepsilon \in [0, \delta_2]$ and $z_1, z_2 \in C(w; \delta_1, \delta_2)$

and write $z_1 = (u_1, \theta_1)^T$ and $z_2 = (u_2, \theta_2)^T$. Then

$$\int_\alpha^{\alpha+\varepsilon} \|(Vz_1)(t) - (Vz_2)(t)\|\, dt \;\leq$$
$$\|\kappa\|_\infty \int_\alpha^{\alpha+\varepsilon} |\theta_1(t)(g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)) - \theta_2(t)(g(t) - (\mathbf{F}_\infty\Phi(u_2))(t))|\, dt$$
$$+ \int_\alpha^{\alpha+\varepsilon} [h(\theta_1(t))|g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| - h(\theta_2(t))|g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)|]\, dt. \quad \text{(A.1)}$$

Now since for any $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$, $2(a_2b_2 - a_1b_1) = (b_1 + b_2)(a_2 - a_1) + (a_1 + a_2)(b_2 - b_1)$ we have

$$|a_1b_1 - a_2b_2| \leq |b_1 + b_2||a_2 - a_1| + |a_1 + a_2||b_2 - b_1|, \quad \forall\, a_1, a_2, b_1, b_2 \in \mathbb{R},$$

and therefore using (A.1)

$$\int_\alpha^{\alpha+\varepsilon} \|(Vz_1)(t) - (Vz_2)(t)\|\, dt \;\leq$$
$$\|\kappa\|_\infty \int_\alpha^{\alpha+\varepsilon} [|\theta_1(t) + \theta_2(t)||(\mathbf{F}_\infty\Phi(u_1))(t)) - (\mathbf{F}_\infty\Phi(u_2))(t)|$$
$$+|\theta_1(t) - \theta_2(t)||2g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)) - (\mathbf{F}_\infty\Phi(u_2))(t)|]\, dt$$
$$+ \int_\alpha^{\alpha+\varepsilon} [|h(\theta_1(t)) - h(\theta_2(t))|(|g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| + |g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)|)$$
$$+|h(\theta_1(t)) + h(\theta_2(t))|||g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| - |g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)||]\, dt. \quad \text{(A.2)}$$

Define $\Theta := \sup_{t\in[0,\alpha]} \|\theta_0(t)\|$. Then since $\theta_1, \theta_2 \in C(\theta_0; \delta_1, \delta_2)$, $u_1, u_2 \in C(u_0; \delta_1, \delta_2)$ and by (3.16)

$$\int_\alpha^{\alpha+\varepsilon} |\theta_1(t) + \theta_2(t)||(\mathbf{F}_\infty\Phi(u_1))(t)) - (\mathbf{F}_\infty\Phi(u_2))(t)|\, dt$$
$$\leq\; 2(\delta_1 + \Theta)\varepsilon\gamma_1 \sup_{\alpha\leq t\leq\alpha+\varepsilon} \|z_1(t) - z_2(t)\|. \quad \text{(A.3)}$$

Define $G := \sup_{t\in[0,\alpha+\delta_2]} \|g(t)\|$. Since $\theta_1, \theta_2 \in C(\theta_0; \delta_1, \delta_2)$, $u_1, u_2 \in C(u_0; \delta_1, \delta_2)$ and by (3.17)

$$\int_\alpha^{\alpha+\varepsilon} |\theta_1(t) - \theta_2(t)||2g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)) - (\mathbf{F}_\infty\Phi(u_2))(t)|\, dt$$
$$\leq\; 2(G + \varepsilon\gamma_1\delta_1 + \sqrt{\varepsilon}\gamma_2)\varepsilon \sup_{\alpha\leq t\leq\alpha+\varepsilon} \|z_1(t) - z_2(t)\|. \quad \text{(A.4)}$$

Since h is locally Lipschitz there exists $\gamma_3 > 0$ such that

$$|h(a_1(t)) - h(a_2(t))| \leq \gamma_3|a_1(t) - a_2(t)|, \quad \forall\, a_1, a_2 \in C(\theta_0; \delta_1, \delta_2), \quad \forall\, t \in [\alpha, \alpha+\delta_2].$$

Therefore, using the fact that $\theta_1, \theta_2 \in C(\theta_0; \delta_1, \delta_2)$,

$$\begin{aligned}\int_\alpha^{\alpha+\varepsilon} & |h(\theta_1(t)) - h(\theta_2(t))|(|g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| + |g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)|)\, dt \\ \leq & \ \gamma_3 \int_\alpha^{\alpha+\varepsilon} |\theta_1(t) - \theta_2(t)|(2|g(t)| + |(\mathbf{F}_\infty\Phi(u_1))(t))| + |(\mathbf{F}_\infty\Phi(u_2))(t)|\, dt\,,\end{aligned}$$

and so, since $u_1, u_2 \in C(u_0; \delta_1, \delta_2)$, using (3.17), we have

$$\begin{aligned}\int_\alpha^{\alpha+\varepsilon} & |h(\theta_1(t)) - h(\theta_2(t))|(|g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| + |g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)|)\, dt \\ \leq & \ \gamma_3 2(G + \varepsilon\gamma_1\delta_1 + \sqrt{\varepsilon}\gamma_2)\varepsilon \sup_{\alpha\leq t\leq\alpha+\varepsilon} \|z_1(t) - z_2(t)\|\,. \qquad \text{(A.5)}\end{aligned}$$

Since $\theta_1, \theta_2 \in C(\theta_0; \delta_1, \delta_2)$, we see that $|h(\theta_1(t)) + h(\theta_2(t))| \leq 2(\gamma_3\delta_1 + h(\theta_0(\alpha)))$ for all $t \in [0, \alpha + \delta_2]$ and therefore using

$$||a + b_1| - |a + b_2|| \leq |b_1 - b_2|\,, \quad \forall\, a, b_1, b_2 \in \mathbb{R}\,,$$

the fact that $u_1, u_2 \in C(u_0; \delta_1, \delta_2)$ and (3.16), we may conclude that

$$\begin{aligned}\int_\alpha^{\alpha+\varepsilon} & |h(\theta_1(t)) + h(\theta_2(t))|||g(t) - (\mathbf{F}_\infty\Phi(u_1))(t)| - |g(t) - (\mathbf{F}_\infty\Phi(u_2))(t)||]\, dt \\ \leq & \ 2(\gamma_3\delta_1 + h(\theta_0(\alpha)))\varepsilon\gamma_1 \sup_{\alpha\leq t\leq\alpha+\varepsilon} \|z_1(t) - z_2(t)\|\,. \qquad \text{(A.6)}\end{aligned}$$

Clearly, (A.2)–(A.6) imply that there exists $\gamma > 0$ such that

$$\int_\alpha^{\alpha+\varepsilon} \|(Vz_1)(t) - (Vz_2)(t)\|\, dt \leq \gamma\varepsilon \sup_{\alpha\leq t\leq\alpha+\varepsilon} \|z_1(t) - z_2(t)\|\,,$$

which implies that V is weakly locally Lipschitz.

Appendix 3: Proof of Lemma 4.1.4

For a set S, let $\#S$ denote the cardinality of S. Fix $t_2 > t_1$ and let $u : [t_1, t_2] \to \mathbb{R}$ be non-decreasing. Define the set

$$\mathfrak{C}_u := \{z \in \operatorname{im} u \,|\, \#u^{-1}(\{z\}) > 1\}\,.$$

If $z \in \mathfrak{C}_u$, then $I_z := u^{-1}(\{z\}) \subset [t_1, t_2]$ is an interval with $\operatorname{int}(I_z) \neq \emptyset$. If u is continuous, then I_z is closed. Moreover,

$$I_{z_1} \cap I_{z_2} = \emptyset\,, \quad \forall\, z_1, z_2 \in \mathfrak{C}_u \text{ with } z_1 \neq z_2. \qquad \text{(A.7)}$$

Before proving Lemma 4.1.4 we prove two auxiliary results.

Lemma A.1 *Let $t_0, t_1, a, b \in \mathbb{R}$ with $t_1 > t_0$ and $b > a$. If $S \subset [a, b]$ is countable, there exists a surjective continuous non-decreasing function $F : [t_0, t_1] \to [a, b]$, such that $S \subset \mathcal{C}_F$.*

Proof: If S is finite, then the lemma is trivially true. Thus without loss of generality we may assume that S is infinite. Since S is countable we may write $S = \{s_i \mid i \in \mathbb{Z}_+\}$ with $s_i \neq s_j$ if $i \neq j$. Without loss of generality we may assume that $a, b \in S$ and that $s_0 = a$ and $s_1 = b$. We recursively define continuous, piecewise linear, non-decreasing, surjective functions $F_j : [t_0, t_1] \to [a, b]$ $(j \in \mathbb{N})$ such that $\mathcal{C}_{F_j} = \{s_0, \ldots, s_j\}$. To this end let $\tau_0, \tau_1 \in (t_0, t_1)$ with $\tau_0 < \tau_1$ and set

$$F_1(t) = \begin{cases} a\,, & t_0 \le t \le \tau_0\,, \\ b\,, & \tau_1 \le t \le t_1\,, \\ a + \frac{b-a}{\tau_1 - \tau_0}(t - \tau_0)\,, & \tau_0 \le t \le \tau_1\,, \end{cases}$$

Suppose that $F_n : [t_0, t_1] \to [a, b]$ is continuous, piecewise linear, non-decreasing and surjective with $\mathcal{C}_{F_n} = \{s_0, \ldots, s_n\}$. Define τ_{n+1} by

$$\{\tau_{n+1}\} := F_n^{-1}(\{s_{n+1}\})$$

and set

$$\tau_{n+1}^l := \min\{t \in [t_1, \tau_{n+1}] \mid F_n \text{ is strictly increasing on } [t, \tau_{n+1}]\}\,,$$

$$\tau_{n+1}^r := \max\{t \in [\tau_{n+1}, t_2] \mid F_n \text{ is strictly increasing on } [\tau_{n+1}, t]\}\,,$$

and

$$l_{n+1} := \frac{1}{2^n} \min\left\{\frac{1}{F_n'(\tau_{n+1})}, \tau_{n+1} - \tau_{n+1}^l, \tau_{n+1}^r - \tau_{n+1}\right\}.$$

We define a function $F_{n+1} : [t_0, t_1] \to [a, b]$ as follows: we set $F_{n+1} = F_n$ on $[t_1, \tau_{n+1}^l] \cup [\tau_{n+1}^r, t_2]$, $F_{n+1} \equiv s_n$ on $[\tau_{n+1} - l_{n+1}, \tau_{n+1} + l_{n+1}]$ and we define F_{n+1} to be affine linear on $[\tau_{n+1}^l, \tau_{n+1} - l_{n+1}]$ and $[\tau_{n+1} + l_{n+1}, \tau_{n+1}^r]$.

The recursively defined functions $F_j : [t_0, t_1] \to [a, b]$ are continuous, piecewise linear, non-decreasing with $F_j(t_0) = a$, $F_j(t_1) = b$ and such that $\mathcal{C}_{F_j} = \{s_0, \ldots, s_j\}$. Additionally, by construction we have

$$\sup_{t \in [t_0, t_1]} |F_{j+1}(t) - F_j(t)| \le \frac{1}{2^j}\,,$$

which implies that the sequence is Cauchy and therefore convergent. We denote the limit by F. By construction $F : [t_0, t_1] \to [a, b]$ is continuous, non-decreasing and such that $F(t_0) = a$, $F(t_1) = b$ and, moreover, $\#F^{-1}(\{s\}) > 1$ for all $s \in S$. $\square$

Lemma A.2 *Let $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0$.*

(1) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing. Then $\mathcal{C}_u$ is at most countable.

(2) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous. If $I \subset [t_0, t_1]$ is an interval with $\operatorname{int}(I) \neq \emptyset$ and $I \subset u^{-1}(\mathcal{C}_u)$, then u is constant on $\operatorname{clos}(I)$.

(3) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous and set

$$D := [t_0, t_1] \setminus u^{-1}(\mathcal{C}_u), \quad E := [u(t_0), u(t_1)] \setminus \mathcal{C}_u .$$

Then $\hat{u} := u|_D$ is a bijective map from D to E and $\hat{u}^{-1} : E \to D$ is continuous.

(4) Let $u, v : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous with $u(t_0) = v(t_0)$ and $u(t_1) = v(t_1)$. If $\mathcal{C}_u \subset \mathcal{C}_v$, then there exists a surjective continuous non-decreasing function $f : [t_0, t_1] \to [t_0, t_1]$, such that $u \circ f = v$.

Proof: (1) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing. As above, for $z \in \mathcal{C}_u$, we set $I_z := u^{-1}(\{z\})$. Choose $q_z \in I_z \cap \mathbb{Q}$ for each $z \in \mathcal{C}_u$, which is possible since $\operatorname{int}(I_z) \neq \emptyset$ for $z \in \mathcal{C}_u$. It follows from (A.7) that the map

$$\mathcal{C}_u \to \mathbb{Q}, \qquad z \mapsto q_z$$

is injective, showing that $\mathcal{C}_u$ is at most countable.

(2) Let $u : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous and $I \subset [t_0, t_1]$ be an interval with $\operatorname{int}(I) \neq \emptyset$ and $I \subset u^{-1}(\mathcal{C}_u)$. Seeking a contradiction, suppose that the claim is not true. Setting $[\alpha, \beta] := \operatorname{clos}(I)$, we have $u(\alpha) < u(\beta)$. Let $z \in (u(\alpha), u(\beta))$. By continuity of u there exists $t \in (\alpha, \beta) \subset I$ such that $u(t) = z$. Hence $z \in u(I)$ and so $z \in \mathcal{C}_u$. Thus

$$(u(\alpha), u(\beta)) \subset \mathcal{C}_u$$

showing that $\mathcal{C}_u$ is uncountable, which is impossible by part (1).

(3) It is clear that $\hat{u} := u|_D : D \to E$ is bijective. It remains to show that $\hat{u}^{-1} : E \to D$ is continuous. Seeking a contradiction, suppose that $\hat{u}^{-1}$ is not continuous. Then there exists $e \in E$, $(e_n) \subset E$ and $\varepsilon > 0$ such that $\lim_{n\to\infty} e_n = e$ and

$$|\hat{u}^{-1}(e_n) - \hat{u}^{-1}(e)| \geq \varepsilon , \quad \forall n \in \mathbb{Z}_+ . \tag{A.8}$$

Setting $d_n := \hat{u}^{-1}(e_n)$ and $d := \hat{u}^{-1}(e)$ we have as $n \to \infty$

$$u(d_n) = \hat{u}(d_n) = e_n \to e = \hat{u}(d) = u(d)\,.$$

Moreover, since $\operatorname{clos}(D)$ is compact, we may assume without loss of generality that $d_n \to d^* \in \operatorname{clos}(D)$ as $n \to \infty$. By (A.18), $|d^* - d| \geq \varepsilon$, and by the continuity of u, $u(d^*) = u(d)$, showing that

$$u(d) \in \mathcal{C}_u\,. \tag{A.9}$$

But by constuction, $d \in D$, implying that

$$u(d) = \hat{u}(d) \in E = [u(t_0), u(t_1)] \setminus \mathcal{C}_u\,,$$

which contradicts (A.9).

(4) Let $u, v : [t_0, t_1] \to \mathbb{R}$ be non-decreasing and continuous with $u(t_0) = v(t_0)$ and $u(t_1) = v(t_1)$ and $\mathcal{C}_u \subset \mathcal{C}_v$. Let $\hat{u}$ be as defined in part (3). Define $\hat{f} : ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \cup \{t_0, t_1\} \to [t_0, t_1]$,

$$\hat{f}(t) := \begin{cases} \hat{u}^{-1}(v(t)) & \text{for } t \in (t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)\,, \\ t_0 & \text{for } t = t_0\,, \\ t_1 & \text{for } t = t_1\,. \end{cases}$$

Obviously, $\hat{f}$ is non-decreasing and, by part (3), $\hat{f}$ is continuous. Now by part (1), there exists $N \subset \mathbb{Z}_+$ such that

$$v^{-1}(\mathcal{C}_u) = \cup_{j \in N} I_j\,,$$

where the $I_j \subset [t_0, t_1]$ are closed intervals with $I_i \cap I_j = \emptyset$ if $i \neq j$. Write $I_j = [a_j, b_j]$. If $a_j \neq t_0$, then for any $\varepsilon > 0$

$$[a_j - \varepsilon, b_j] \cap ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \neq \emptyset\,,$$

since otherwise there would exist $\varepsilon > 0$ such that

$$[a_j - \varepsilon, b_j] \subset v^{-1}(\mathcal{C}_u) \subset v^{-1}(\mathcal{C}_v)\,,$$

and hence, by part (2), $v(t) = v(a_j)$ for all $t \in [a_j - \varepsilon, b_j]$, which is impossible, since $v^{-1}(\{v(a_j)\}) = [a_j, b_j]$.

Similarly, if $b_j \neq t_1$, then for any $\varepsilon > 0$

$$[a_j, b_j + \varepsilon] \cap ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \neq \emptyset\,.$$

Therefore, we may define

$$\alpha_j := \begin{cases} \lim_{t\uparrow a_j} \hat{f}(t) & \text{if } a_j \neq t_0\,, \\ t_0 & \text{if } a_j = t_0\,, \end{cases} \qquad \text{and} \qquad \beta_j := \begin{cases} \lim_{t\downarrow b_j} \hat{f}(t) & \text{if } b_j \neq t_1\,, \\ t_1 & \text{if } b_j = t_1\,. \end{cases}$$

The limits on the right-hand side exist and are finite since $\hat{f}$ is bounded and non-decreasing. Finally, we define $f : [t_0, t_1] \to [t_0, t_1]$ by

$$f(t) = \begin{cases} \hat{f}(t) & \text{for } t \in ((t_0, t_1) \setminus v^{-1}(\mathcal{C}_u)) \cup \{t_0, t_1\}\,, \\ \alpha_j & \text{for } t = a_j \neq t_0\,, \\ \beta_j & \text{for } t = b_j \neq t_1\,, \\ \alpha_j + \frac{\beta_j - \alpha_j}{b_j - a_j}(t - a_j) & \text{for } t \in (a_j, b_j)\,. \end{cases}$$

By constuction, f is continuous, non-decreasing with $f(t_0) = t_0$, $f(t_1) = t_1$ and $u \circ f = v$. □

Proof of Lemma 4.1.4: Note that

$$R(u) = R(u \circ h)\,, \quad \forall\, u \in C^{\text{uc}}_{\text{pm}}(\mathbb{R}_+, \mathbb{R})\,, \quad \forall\, h \in \mathcal{T}\,. \tag{A.10}$$

This follows from the fact that if $0 = t_0 < t_1 < \ldots < t_n$ is the standard monotonicity partition of $u \circ h$, then $0 = h(t_0) < h(t_1) < \ldots < h(t_n)$ is the standard monotonicity partition of u. Let $u, v \in C^{\text{uc}}_{\text{pm}}(\mathbb{R}_+, \mathbb{R})$. We first assume that there exist $f, g \in \mathcal{T}$ such that $u \circ f = v \circ g$. Then, using (A.10), $R(u) = R(u \circ f) = R(v \circ g) = R(v)$, as required.

To prove the converse, assume that $R(u) = R(v)$. By (A.10), without loss of generality we may assume that u and v have the same standard monotonicity partition $0 = t_0 < t_1 < \ldots < t_n$. For $i \in \{0, 1, \ldots, n-1\}$ we define $I_i := [t_i, t_{i+1}]$. Without loss of generality we may assume that both u and v are non-decreasing on I_i. Define $u_i := u|_{I_i}$, $v_i := v|_{I_i}$ and $J_i := [u(t_i), u(t_{i+1})] = [v(t_i), v(t_{i+1})]$.

By Lemma A.2, part (1), $\mathcal{C}_{u_i} \cup \mathcal{C}_{v_i}$ is at most countable and so by Lemma A.1 there exists a surjective continuous non-decreasing function $F_i : I_i \to J_i$, with $\mathcal{C}_{u_i} \cup \mathcal{C}_{v_i} \subset \mathcal{C}_{F_i}$. Therefore by Lemma A.2, part (4), there exist non-decreasing continuous functions $f_i, g_i : I_i \to I_i$ with $f_i(t_i) = t_i$, $f_i(t_{i+1}) = t_{i+1}$, $g_i(t_i) = t_i$ and $g_i(t_{i+1}) = t_{i+1}$ and such that $u_i \circ f_i = F_i = v_i \circ g_i$.

Define functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(t) = \begin{cases} f_i(t) & \text{if } t \in I_i\,, \\ t & \text{if } t \geq t_n\,, \end{cases}$$

and

$$g(t) = \begin{cases} g_i(t) & \text{if } t \in I_i\,, \\ t & \text{if } t \geq t_n\,. \end{cases}$$

By construction $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, non-decreasing and such that $f(0) = g(0) = 0$, $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty$ and $u \circ f = v \circ g$.

□

Appendix 4: Derivation of (4.36)

First we prove a simple lemma.

Lemma A.3 *Let $\Phi : C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ be a hysteresis operator and let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$. Assume that $\tau > 0$ is such that u is not left-continuous at τ. Let $0 < \tau_1 < \tau_2 < \ldots < \tau_n = \tau$ denote the points of discontinuity of $\mathbf{Q}_\tau u$, set $\tau_0 = 0$ and define ε_k by (4.32). Then, for all sufficiently large k*

$$(\Phi(C_k(\mathbf{Q}_\tau u)))(\tau - \varepsilon_k/2) = (\tilde{\Phi}(u))(\tau-)\,.$$

Proof: Define $v \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by

$$v(t) = \begin{cases} u(t) & \text{if } t \in [0, \tau)\,, \\ u(\tau-) & \text{if } t \geq \tau\,. \end{cases}$$

Set $s_k := \tau - \varepsilon_k/2$ and $w_k := \mathbf{Q}_{s_k} C_k(\mathbf{Q}_\tau u)$. There exists $l_1 > 0$ such that

$$R(\mathbf{Q}_\tau C_k(v)) = R(C_k(v)) = R(w_k) = R(\mathbf{Q}_\tau w_k)\,, \quad \forall\, k \geq l_1\,. \tag{A.11}$$

Moreover, by Lemma 4.4.6, statement (1), there exists $l_2 > 0$ such that

$$\tilde{R}(\mathbf{Q}_\tau v) = R(\mathbf{Q}_\tau C_k(v))\,, \quad \forall\, k \geq l_2\,. \tag{A.12}$$

Setting $l := \max(l_1, l_2)$, it follows from (A.11) and (A.12) that for all $k \geq l$, $\tilde{R}(\mathbf{Q}_\tau v) = R(\mathbf{Q}_\tau w_k)$. Hence, since v is left-continuous at τ we may conclude using Theorem 4.1.2, Theorem 4.4.5 and Corollary 4.4.10 that for all $k \geq l$

$$\begin{aligned} (\tilde{\Phi}(u))(\tau-) &= (\tilde{\Phi}(v))(\tau) = (\Phi(w_k))(\tau) \\ &= (\Phi(\mathbf{Q}_{s_k} C_k(\mathbf{Q}_\tau u)))(\tau) = (\Phi(C_k(\mathbf{Q}_\tau u)))(s_k)\,. \end{aligned}$$

□

Let $u \in NPC_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ and let $0 = t_0 < t_1 < t_2 < \ldots$ be such that $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each of the intervals (t_i, t_{i+1}). It is clear that $(\tilde{\mathfrak{B}}_{h,\xi}(u))(0)$

$= b_h(u(0), \xi)$. Let $t > t_0 = 0$; then there exists $i \in \mathbb{N}$ such that $t \in (0, t_1) \cup [t_i, t_{i+1})$. There exists $l > 0$ such that

$$(C_k(\mathbf{Q}_t u))(t) = u(t), \quad \forall k \geq l, \tag{A.13}$$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(\tau) = (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_\tau u)))(\tau), \quad \forall k \geq l, \quad \forall \tau \in \{t_i, t\}, \tag{A.14}$$

where (A.14) follows from Proposition 4.4.7, part (1). We consider three cases.

CASE 1. Suppose that $t \in (0, t_1)$.
Clearly, u is monotone on $[0, t]$ and so is $C_k(Q_t u)$. Hence, by (A.13) and (A.14)

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(0)) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(0)).$$

CASE 2. Suppose that $t = t_i$.
There are two subcases which need to be distinguished.
SUBCASE A. u is left-continuous at $t = t_i$.
There exists $\varepsilon > 0$ such that u is monotone on $[t - \varepsilon, t]$. It follows that there exists $l_1 \geq l$ such that $C_k(\mathbf{Q}_t u)$ is monotone on $[t - \varepsilon, t]$ for all $k \geq l_1$. Choose $(s_n) \subset [t - \varepsilon, t)$ with $\lim_{n\to\infty} s_n = t$. Using (A.14) and Corollary 4.4.10, we have

$$(\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(t) = (\tilde{\mathcal{B}}_{h,\xi}(u))(t) = (\tilde{\mathcal{B}}_{h,\xi}(u))(t-), \quad \forall k \geq l_1. \tag{A.15}$$

Using (A.13) and (A.14) it follows for $k \geq l_1$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(s_n)),$$

and hence, since $\lim_{n\to\infty}(\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(s_n) = (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(t)$

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(t)). \tag{A.16}$$

Therefore, combining (A.15) and (A.16) gives

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(t-)).$$

SUBCASE B. u is not left-continuous at $t = t_i$.
Define for each $k \in \mathbb{Z}_+$, $s_k := t - \varepsilon_k/2$, where ε_k is defined by (4.32). $C_k(\mathbf{Q}_t u)$ is monotone on $[s_k, t]$ and therefore using (A.13), (A.14) and Lemma A.3, for all sufficiently large k

$$(\tilde{\mathcal{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathcal{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(s_k)) = b_h(u(t), (\tilde{\mathcal{B}}_{h,\xi}(u))(t-)).$$

CASE 3. $t \in (t_i, t_{i+1})$

We consider the same two subcases as in Case 2.

SUBCASE A. u is left-continuous at t_i.

Define for each $k \in \mathbb{Z}_+$, $s_k := t_i + \varepsilon_k/2$, where ε_k is defined by (4.32). There exists $l_1 \geq l$ such that for $k \geq l_1$, $C_k(\mathbf{Q}_t u)$ is monotone on $[s_k, t]$. Therefore, using (A.13) and (A.14)

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathfrak{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(s_k)), \quad \forall k \geq l_1. \tag{A.17}$$

Moreover, for all sufficiently large k, $C_k(\mathbf{Q}_t u)$ is monotone on $[t_i, s_k]$. Hence, using (A.14) and Corollary 4.4.10, we may conclude that for all sufficiently large k

$$\begin{aligned}(\mathfrak{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(s_k) &= b_h((C_k(\mathbf{Q}_t u))(s_k), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i)) \\ &= b_h((C_k(\mathbf{Q}_t u))(s_k), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-)).\end{aligned}$$

Combining this with (A.17) gives, for all sufficiently large k

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = b_h(u(t), b_h((C_k(\mathbf{Q}_t u))(s_k), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-))).$$

Since $\lim_{k\to\infty}(C_k(\mathbf{Q}_t u))(s_k) = u(t_i+)$, we have

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = b_h(u(t), b_h(u(t_i+), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-))).$$

SUBCASE B. u is not left-continuous at t_i.

Therefore, u is right-continuous at t_i and hence u is monotone on $[t_i, t]$ and so is $C_k(\mathbf{Q}_t u)$ for all sufficiently large k. By (A.13) and (A.14), for all sufficiently large k

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = b_h(u(t), (\mathfrak{B}_{h,\xi}(C_k(\mathbf{Q}_t u)))(t_i)) = b_h(u(t), (\mathfrak{B}_{h,\xi}(C_k(\mathbf{Q}_{t_i} u)))(t_i)).$$

Applying (A.14) again, we obtain

$$(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) = b_h(u(t), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i)),$$

and so, by Case 2,

$$\begin{aligned}(\tilde{\mathfrak{B}}_{h,\xi}(u))(t) &= b_h(u(t), b_h(u(t_i), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-))) \\ &= b_h(u(t), b_h(u(t_i+), (\tilde{\mathfrak{B}}_{h,\xi}(u))(t_i-))).\end{aligned}$$

Appendix 5: Constuction of the counterexample mentioned in Remark 5.3.1, part (2).

We show, by constructing a counterexample, that the numerical value set of a nonlinearity $\Phi \in \mathcal{N}_d(\lambda)$ is not necessarily an interval. It is convenient to prove a technical lemma first.

For all $x \in \mathbb{R}_+$, define $\lfloor x \rfloor$ to be the largest integer such that $\lfloor x \rfloor \leq x$. For each $x \in \mathbb{R}_+$ there exist numbers $n_i(x) \in \{0, 1, \dots, 9\}$, $i \in \mathbb{N}$, such that

$$x = \lfloor x \rfloor + \sum_{i=1}^{\infty} 10^{-i} n_i(x) .$$

For all $k \in \mathbb{N}$, define $\tilde{f}_k : \mathbb{R}_+ \to \mathbb{Q} \cap \mathbb{R}_+$ by

$$\tilde{f}_k(x) := \lfloor x \rfloor + \sum_{i=1}^{k} 10^{-i} n_i(x) , \quad \forall x \in \mathbb{R}_+ .$$

Moreover, we define for all $k \in \mathbb{N}$, the truncation function $f_k : \mathbb{R} \to \mathbb{Q}$ by

$$f_k(x) := \begin{cases} \tilde{f}_k(x) & \text{for } x \geq 0 , \\ -\tilde{f}_k(-x) & \text{for } x < 0 . \end{cases}$$

Lemma A.4

(1) $f_k(0) = 0$ *for all* $k \in \mathbb{N}$*;*

(2) $f_k(-x) = -f_k(x)$ *for all* $x \in \mathbb{R}$ *and all* $k \in \mathbb{N}$*;*

(3) $f_k(x)/x \in [0, 1]$ *for all* $x \in \mathbb{R} \setminus \{0\}$ *and all* $k \in \mathbb{N}$*;*

(4) $x \leq f_k(x) + 10^{-k}$ *for all* $x \in \mathbb{R}$ *and all* $k \in \mathbb{N}$.

Proof: Statements (1) and (2) follow immediately from the definitions of $\tilde{f}_k$ and f_k. To prove (3), let $k \in \mathbb{N}$ and assume that $x > 0$. Then

$$\frac{f_k(x)}{x} = \frac{\tilde{f}_k(x)}{x} = \frac{\lfloor x \rfloor + \sum_{i=1}^{k} 10^{-i} n_i(x)}{\lfloor x \rfloor + \sum_{i=1}^{\infty} 10^{-i} n_i(x)} \in [0, 1] . \tag{A.18}$$

If $x < 0$, then by statement (2) and (A.18)

$$\frac{f_k(x)}{x} = \frac{f_k(-x)}{-x} \in [0, 1] .$$

To prove statement (4), note that for all $x \geq 0$ and all $k \in \mathbb{N}$

$$0 \leq x - f_k(x) = \sum_{i=k+1}^{\infty} 10^{-i} n_i(x) \leq 9 \sum_{i=k+1}^{\infty} 10^{-i} = 10^{-k}.$$

If $x < 0$, then by statement (2), for all $k \in \mathbb{N}$

$$\begin{aligned} x &= -\lfloor -x \rfloor - \sum_{i=1}^{\infty} 10^{-i} n_i(-x) \leq -\lfloor -x \rfloor - \sum_{i=1}^{k} 10^{-i} n_i(-x) \\ &= -f_k(-x) = f_k(x) \leq f_k(x) + 10^{-k}. \end{aligned}$$

□

Consider the operator $\Phi : F(\mathbb{Z}_+, \mathbb{R}) \to F(\mathbb{Z}_+, \mathbb{R})$ defined by

$$(\Phi(u))(n) = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{k=1}^{n} f_k(u(k) - u(k-1)) & \text{for } n \in \mathbb{N}. \end{cases}$$

Obviously, $\mathrm{NVS}\,\Phi \subset \mathbb{Q}$, showing that $\mathrm{NVS}\,\Phi$ is not an interval. We prove that $\Phi \in \mathcal{N}_d(1)$. It is immediately clear that Φ satisfies (D1). Since

$$(\Phi(u))(n+1) = (\Phi(u))(n) + f_{n+1}(u(n+1) - u(n)), \quad \forall\, u \in F(\mathbb{N}, \mathbb{R}), \quad \forall\, n \in \mathbb{Z}_+,$$

we have for all $u \in F(\mathbb{Z}_+, \mathbb{R})$ and all $n \in \mathbb{Z}_+$ such that $u(n+1) - u(n) \neq 0$

$$\frac{(\Phi(u))(n+1) - (\Phi(u))(n)}{u(n+1) - u(n)} = \frac{f_{n+1}(u(n+1) - u(n))}{u(n+1) - u(n)} \in [0, 1],$$

where we have used Lemma A.4, part (3). Therefore (D2) is satisfied for $\lambda = 1$.

For (D3), let $u \in F(\mathbb{Z}_+, \mathbb{R})$ be ultimately non-decreasing and $\lim_{n\to\infty} u(n) = \infty$. Hence, $\Phi(u)$ is ultimately non-decreasing, showing that $L := \lim_{n\to\infty}(\Phi(u))(n)$ exists and $L \in \mathbb{R} \cup \{\infty\}$. Clearly $\sup \mathrm{NVS}\,\Phi = \infty$ and we have to show that $\lim_{n\to\infty}(\Phi(u))(n) = \infty$. Seeking a contradiction, suppose that $\lim_{n\to\infty}(\Phi(u))(n) \neq \infty$; then, $\lim_{n\to\infty}(\Phi(u))(n) = L \in \mathbb{R}$, and so

$$\sum_{k=1}^{\infty} f_k(u(k) - u(k-1)) = L < \infty. \tag{A.19}$$

Using Lemma A.4, part (4), for all $n \in \mathbb{N}$

$$u(n) - u(0) = \sum_{k=1}^{n} (u(k) - u(k-1)) \leq \sum_{k=1}^{n} f_k(u(k) - u(k-1)) + \sum_{k=1}^{n} 10^{-k}. \tag{A.20}$$

Combining (A.20) with (A.19), we may conclude that u is bounded above, which is in contradiction to $\lim_{n\to\infty} u(n) = \infty$. From Lemma A.4, part (2) and the definition of Φ, we see that $(\Phi(-u))(n) = -(\Phi(u))(n)$ for all $n \in \mathbb{Z}_+$. Therefore, $\lim_{n\to\infty}(\Phi(-u))(n) = -\lim_{n\to\infty}(\Phi(u))(n) = -\infty = \inf \mathrm{NVS}\,\Phi$.

For (D4), let $u \in F(\mathbb{Z}_+, \mathbb{R})$ and suppose that $L := \lim_{n\to\infty}(\Phi(u))(n)$ exists and $L \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi)) = \mathbb{R}$. Thus (A.19) is satisfied. Combining (A.19) and (A.20), we may again conclude that u is bounded above. Since $\lim_{n\to\infty}(\Phi(-u))(n) = -\lim_{n\to\infty}(\Phi(u))(n) = -L \in \operatorname{int}(\operatorname{clos}(\mathrm{NVS}\,\Phi))$, the above argument can be applied to $-u$ and we may conclude that $-u$ is also bounded above, implying that u is bounded below.

Appendix 6: Proof that for the diffusion example $\mathfrak{L}^{-1}(\mathbf{G}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$ for any $\alpha > -\kappa\pi^2$

Let us first consider

$$\tilde{\mathbf{G}}(s) = \frac{\sinh\left(x_1\sqrt{s/\kappa}\right)\sinh\left((1-x_2)\sqrt{s/\kappa}\right)}{\kappa\sqrt{s/\kappa}\sinh\sqrt{s/\kappa}},$$

where $\kappa > 0$ and $0 < x_1 < x_2 < 1$. For $\theta \in (0, \pi)$ set

$$\mathcal{S}_\theta := \{s \in \mathbb{C} \setminus \{0\} \mid |\arg s| < \theta\}.$$

We first prove a lemma which was communicated by H. Logemann.

Lemma A.5 *Let $\delta \in (0, \pi/2)$, then*

$$\lim_{|s|\to\infty,\, s\in\mathcal{S}_{\frac{\pi}{2}+\delta}} |s\tilde{\mathbf{G}}(s)| = 0.$$

Proof: For notational convenience set $z := \sqrt{s/\kappa}$. For $s \in \mathcal{S}_{\pi/2+\delta}$

$$\begin{aligned} s\tilde{\mathbf{G}}(s) &= z^2\kappa\frac{\sinh(x_1 z)\sinh((1-x_2)z)}{\kappa z\sinh z} \\ &= z\frac{(e^{x_1 z} - e^{-x_1 z})(e^{(1-x_2)z} - e^{-(1-x_2)z})}{2(e^z - e^{-z})} \\ &= z\frac{e^{(x_1-x_2)z} - e^{-(x_1+x_2)z} - e^{(-2+x_1+x_2)z} + e^{(-2+x_2-x_1)z}}{2(1-e^{-2z})}. \end{aligned}$$

Since $0 < x_1 < x_2 < 1$, there exists $\alpha > 0$ such that

$$|s\tilde{\mathbf{G}}(s)| \le \frac{|z|}{2}\frac{e^{-\alpha \operatorname{Re} z}}{1 - e^{-2\operatorname{Re} z}}, \quad \forall\, s \in \mathcal{S}_{\pi/2+\delta}. \tag{A.22}$$

Now $\pi/4 + \delta/2 < \pi/2$ and hence, since $z \in \mathbb{S}_{\pi/4+\delta/2} \subset \mathbb{C}_0$, there exists $M > 0$ such that

$$|z| \le M \operatorname{Re} z\,, \quad \forall\, s \in \mathbb{S}_{\pi/2+\delta}\,.$$

Hence, by (A.22),

$$|s\tilde{\mathbf{G}}(s)| \le \frac{M \operatorname{Re} z}{2} \frac{e^{-\alpha \operatorname{Re} z}}{1 - e^{-2\operatorname{Re} z}}\,, \quad \forall\, s \in \mathbb{S}_{\pi/2+\delta}\,.$$

As $|s| \to \infty$ in $\mathbb{S}_{\pi/2+\delta}$, $|z| \to \infty$ in $\mathbb{S}_{\pi/4+\delta/2}$ and so

$$\lim_{|s|\to\infty,\, s \in \mathbb{S}_{\pi/2+\delta}} |s\tilde{\mathbf{G}}(s)| = 0\,.$$

□

The Hardy space, of order 2, of holomorphic functions defined on $\mathbb{C}_\alpha$ is denoted by $H^2(\mathbb{C}_\alpha)$. A holomorphic function $f : \mathbb{C}_\alpha \to \mathbb{C}$ is an element of $H^2(\mathbb{C}_\alpha)$ if

$$\sup_{x>\alpha} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dy < \infty\,.$$

Let $\alpha > -\kappa\pi^2$ and $\beta \in (-\kappa\pi^2, \alpha)$. To show that $\mathfrak{L}^{-1}(\tilde{\mathbf{G}}) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$, we first show that $\tilde{\mathbf{G}} \in H^2(\mathbb{C}_\beta)$. Note that $\tilde{\mathbf{G}} \in H^\infty(\mathbb{C}_\beta)$. An application of Lemma A.5 shows that

$$\lim_{|s|\to\infty,\, s \in \mathbb{C}_\beta} |s\tilde{\mathbf{G}}(s)| = 0\,.$$

We may conclude that there exists $M > 0$ such that

$$|\tilde{\mathbf{G}}(x+iy)|^2 \le \frac{M}{y^2+1}\,, \quad \forall\, x > \beta\,, \quad \forall\, y \in \mathbb{R}\,,$$

and so

$$\sup_{x>\beta} \int_{-\infty}^{\infty} |\tilde{\mathbf{G}}(x+iy)|^2\, dy < \infty\,.$$

Hence $\tilde{\mathbf{G}} \in H^2(\mathbb{C}_\beta)$ and by a well-known theorem of Paley and Wiener, $\mathfrak{L}^{-1}(\tilde{\mathbf{G}}) \in L^2_\beta(\mathbb{R}_+, \mathbb{R}) \subset L^1_\alpha(\mathbb{R}_+, \mathbb{R}) \subset \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$.

Since $\mathbf{G}(s) = e^{-sT}\tilde{\mathbf{G}}(s)$, we have $\mathfrak{L}^{-1}(\mathbf{G}) = \mathbf{R}_T(\mathfrak{L}^{-1}(\tilde{\mathbf{G}})) \in \mathcal{M}_{\mathrm{f}}^{\alpha}(\mathbb{R}_+)$.

Appendix 7: The discrete-time positive real lemma

The following result is a version of the discrete-time infinite-dimensional positive real lemma.

Lemma A.6 *For a real Hilbert space X, let $A^d \in L(X)$, $B^d \in L(\mathbb{R}, X)$, $C^d \in L(X, \mathbb{R})$ and $D^d \in \mathbb{R}$ and set $\mathbf{G}^d(z) := C^d(zI - A^d)^{-1}B^d + D^d$. Assume that A^d is power-stable and*

$$\operatorname{Re}\mathbf{G}^d(e^{i\theta}) > 0\,, \quad \forall\, \theta \in [0, 2\pi)\,.$$

Then there exist $P^d \in L(X)$, $P^d = (P^d)^* \geq 0$, $L^d \in L(\mathbb{R}, X)$ and $W^d \in \mathbb{R}$ such that

$$\begin{aligned} (A^d)^* P^d A^d - P^d &= -L^d (L^d)^*\,, && \text{(A.23a)} \\ (A^d)^* P^d B^d &= (C^d)^* - L^d W^d\,, && \text{(A.23b)} \\ (W^d)^2 &= 2D^d - (B^d)^* P^d B^d\,. && \text{(A.23c)} \end{aligned}$$

Although Lemma A.6 should be well-known, we were not able to locate it in the literature. To prove Lemma A.6 we make use of an infinite-dimensional version of the continuous-time positive real lemma stated below.

Lemma A.7 *For a real Hilbert space X, let $A \in L(X)$, $B \in L(\mathbb{R}, X)$, $C \in L(X, \mathbb{R})$ and $D \in \mathbb{R}$, let $\sigma(A)$ denote the spectrum of A and set $\mathbf{G}(s) := C(sI - A)^{-1}B + D$. Assume that $\sigma(A) \subset \{s \in \mathbb{C} \,|\, \operatorname{Re} s < 0\}$ and*

$$\operatorname{Re}\mathbf{G}(i\omega) > 0\,, \quad \forall\, \omega \in \mathbb{R} \cup \{\pm\infty\}\,. \tag{A.24}$$

Then there exist $P \in L(X)$, $P = P^* \geq 0$, $L \in L(\mathbb{R}, X)$ and $W > 0$ such that

$$\begin{aligned} PA + A^*P &= -LL^*\,, && \text{(A.25a)} \\ PB &= C^* - LW\,, && \text{(A.25b)} \\ 2D &= W^2\,. && \text{(A.25c)} \end{aligned}$$

In a different form, Lemma A.7 is due to Yakubovich [47] (see also Wexler [46]). For completeness we include a proof which is based on the positive-real Riccati equation theory developed in van Keulen [15].

Proof of Lemma A.7: By (A.24) we have that $D > 0$; defining $W := \sqrt{2D}$ gives (A.25c). Furthermore, again by (A.24), it follows from [15] (see Theorem 3.10 and Remark 3.14 in [15]), that there exists $Q \in L(X)$, $Q = Q^*$, such that

$$QA + A^*Q = (1/W)^2 (B^*Q + C)^* \, (B^*Q + C)\,,$$

Setting

$$P := -Q\,, \quad L := (1/W)(C - B^*P)^*$$

yields (A.25a) and (A.25b). Since $\sigma(A) \subset \{s \in \mathbb{C} \,|\, \operatorname{Re} s < 0\}$, it follows from (A.25a) by a routine argument that $P \geq 0$. □

The following proof of Lemma A.6 (suggested by H. Logemann) makes use standard fractional transformation techniques (as used in [14] for the finite-dimensional case).

Proof of Lemma A.6: Define

$$\begin{aligned} A &:= (A^d+I)^{-1}(A^d-I) \in L(X), \\ B &:= 2(A^d+I)^{-2}B^d \in L(\mathbb{R}, X), \\ C &:= C^d \in L(X, \mathbb{R}), \\ D &:= D^d - C^d(A^d+I)^{-1}B^d = \mathbf{G}^d(-1) \in \mathbb{R}, \end{aligned}$$

and set $\mathbf{G}(s) = C(sI-A)^{-1}B + D$. We proceed in three steps.

STEP 1. We claim that there exists $\varepsilon > 0$ such that

$$\sigma(A) \subset \mathbb{C}_{-\varepsilon}, \tag{A.26}$$

where $\sigma(A)$ denotes the spectrum of A. To this end, introduce the function

$$f(z) = \frac{z-1}{z+1}.$$

f is holomorphic on $\mathbb{E}_1 \supset \sigma(A^d)$ and

$$A = (A^d+I)^{-1}(A^d-I) = f(A^d).$$

By the spectral mapping theorem (see Theorem 48.2, pp. 202–203, in [13]) we have that

$$\sigma(A) = \sigma(f(A^d)) = f(\sigma(A^d)) = \left\{ \frac{z-1}{z+1} \mid z \in \sigma(A^d) \right\}. \tag{A.27}$$

By the power-stability of A^d there exists $\rho \in (0,1)$ such that

$$\sigma(A^d) \subset \mathbb{E}_\rho.$$

Therefore it follows from (A.27) that there exists $\varepsilon > 0$ such that (A.26) holds (for example $\varepsilon = (1-\rho)/4$ would be a suitable choice).

STEP 2. We claim that (A.24) holds. To show that (A.24) holds it is sufficient to prove that

$$\mathbf{G}(s) = \mathbf{G}^d\left(\frac{1+s}{1-s}\right). \tag{A.28}$$

To this end note that

$$\begin{aligned}
\mathbf{G}^d\left(\frac{1+s}{1-s}\right) &= (1-s)C^d((1+s)I-(1-s)A^d)^{-1}B^d + D^d \\
&= (1-s)C^d((I-A^d)+s(I+A^d))^{-1}B^d + D^d \\
&= (1-s)C^d(sI-A)^{-1}(I+A^d)^{-1}B^d + D^d \\
&= C^d(sI-A)^{-1}(I+A^d)^{-1}B^d + D^d \\
&\qquad -C^d(I+(sI-A)^{-1}A)(I+A^d)^{-1}B^d \\
&= D + C^d(sI-A)^{-1}[I-(A^d+I)^{-1}(A^d-I)](I+A^d)^{-1}B^d \\
&= D + C^d(sI-A)^{-1}[A^d+I-(A^d-I)](I+A^d)^{-2}B^d \\
&= C(sI-A)^{-1}B + D = \mathbf{G}(s).
\end{aligned}$$

STEP 3. Step 1 and Step 2 show that the assumptions of Lemma A.6 hold and therefore there exist $P \in L(X)$, $P = P^* \geq 0$, $L \in L(\mathbb{R}, X)$ and $W > 0$ such that (A.25) holds. Define

$$\begin{aligned}
P^d &:= 2((A^d)^* + I)^{-1}P(A^d+I)^{-1}, \\
L^d &:= L, \\
W^d &:= W + L^*(A^d+I)^{-1}B^d.
\end{aligned}$$

Clearly, $P^d = (P^d)^* \geq 0$. We claim that P^d, L^d and W^d satisfy (A.23). By (A.25a)

$$P(A^d+I)^{-1}(A^d-I) + ((A^d)^*-I)((A^d)^*+I)^{-1}P = -LL^*,$$

which implies that

$$((A^d)^*+I)P^d(A^d-I) + ((A^d)^*-I)P^d(A^d+I) = -2L^d(L^d)^*.$$

Therefore,

$$2(A^d)^*P^dA^d - 2P^d = -2L^d(L^d)^*,$$

which yields (A.23a). By (A.25b)

$$2P(A^d+I)^{-2}B^d = (C^d)^* - L^d(W^d - (L^d)^*(A^d+I)^{-1}B^d),$$

which implies that

$$((A^d)^*+I)P^d(A^d+I)^{-1}B^d = (C^d)^* - L^dW^d + L^d(L^d)^*(A^d+I)^{-1}B^d.$$

Therefore,

$$[(A^d)^*P^d + P^d - L^d(L^d)^*](A^d+I)^{-1}B^d = (C^d)^* - L^dW^d\,. \tag{A.30}$$

By (A.23a)

$$P^d - L^d(L^d)^* = (A^d)^*P^dA^d\,,$$

and so, by (A.30),

$$(A^d)^*P^dB^d = (C^d)^* - L^dW^d\,,$$

which is (A.23b). Finally, (A.25c) gives

$$(W^d - (L^d)^*(A^d+I)^{-1}B^d)^2 = 2D^d - 2C^d(A^d+I)^{-1}B^d\,,$$

and so

$$\begin{aligned}(W^d)^2 \;&=\; (B^d)^*((A^d)^*+I)^{-1}L^dW^d + W^d(L^d)^*(A^d+I)^{-1}B^d \\ &\quad -(B^d)^*((A^d)^*+I)^{-1}L^d(L^d)^*(A^d+I)^{-1}B^d \\ &\quad +2D^d - 2C^d(A^d+I)^{-1}B^d\,. \end{aligned} \tag{A.31}$$

Using (A.23a) and (A.23b) combined with (A.31), we have

$$\begin{aligned}(W^d)^2 \;&=\; 2D^d - (B^d)^*((A^d)^*+I)^{-1}[((A^d)^*+I)P^d + P^d(A^d+I) \\ &\quad +(A^d)^*P^dA^d - P^d](A^d+I)^{-1}B^d \\ &=\; 2D^d - (B^d)^*((A^d)^*+I)^{-1}[(A^d)^*P^d + ((A^d)^*+I)P^dA^d + P^d] \\ &\quad (A^d+I)^{-1}B^d \\ &=\; 2D^d - (B^d)^*((A^d)^*+I)^{-1}[((A^d)^*+I)P^d + ((A^d)^*+I)P^dA^d] \\ &\quad (A^d+I)^{-1}B^d \\ &=\; 2D^d - (B^d)^*P^dB^d\,, \end{aligned}$$

which is (A.23c). □

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